

# $L^2$ CASTELNUOVO-DE FRANCHIS, THE CUP PRODUCT LEMMA, AND FILTERED ENDS OF KÄHLER MANIFOLDS

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**ABSTRACT.** Simple approaches to the proofs of the  $L^2$  Castelnuovo-de Franchis theorem and the cup product lemma which give new versions are developed. For example, suppose  $\omega_1$  and  $\omega_2$  are two linearly independent closed holomorphic 1-forms on a bounded geometry connected complete Kähler manifold  $X$  with  $\omega_2$  in  $L^2$ . According to a version of the  $L^2$  Castelnuovo-de Franchis theorem obtained in this paper, if  $\omega_1 \wedge \omega_2 \equiv 0$ , then there exists a surjective proper holomorphic mapping of  $X$  onto a Riemann surface for which  $\omega_1$  and  $\omega_2$  are pull-backs. Previous versions required both forms to be in  $L^2$ .

## INTRODUCTION

According to the classical theorem of Castelnuovo and de Franchis (see [Be], [BarPV]), if, on a connected compact complex manifold  $X$ , there exist linearly independent closed holomorphic 1-forms  $\omega_1$  and  $\omega_2$  with  $\omega_1 \wedge \omega_2 \equiv 0$ , then there exist a surjective holomorphic mapping  $\Phi$  of  $X$  onto a curve  $C$  of genus  $g \geq 2$  and holomorphic 1-forms  $\theta_1$  and  $\theta_2$  on  $C$  such that  $\omega_j = \Phi^*\theta_j$  for  $j = 1, 2$ . The main point is that the meromorphic function  $f \equiv \omega_1/\omega_2$  actually has no points of indeterminacy, so one may Stein factor the holomorphic map  $f: X \rightarrow \mathbb{P}^1$ .

*Remark.* The requirement that the forms be closed is superfluous if the compact manifold  $X$  is a surface or if  $X$  is Kähler. For, if  $\eta = \sum \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$  is the Kähler form for a Kähler metric  $g$  and  $\omega$  is a holomorphic 1-form, then, by Stokes' theorem, we have

$$\int_X d\omega \wedge d\bar{\omega} \wedge \eta^{n-2} = 0;$$

where  $n = \dim X$ . Since the integrand is a nonnegative  $2n$ -form, the form must vanish and it follows that  $d\omega = 0$ . For  $X$  a surface, the same argument with the factor  $\eta^{n-2}$  removed again yields  $d\omega = 0$ .

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In general, given a connected complex manifold  $X$  and linearly independent closed holomorphic 1-forms  $\omega_1$  and  $\omega_2$  on  $X$  with  $\omega_1 \wedge \omega_2 \equiv 0$ , the meromorphic function  $f \equiv \omega_1/\omega_2$  has no points of indeterminacy,  $f$  is locally constant on the analytic set  $Z = \{x \in X \mid (\omega_1)_x = 0 \text{ or } (\omega_2)_x = 0\}$ , and  $f$  is constant on each leaf of the holomorphic foliation determined by  $\omega_1$  and  $\omega_2$  in  $X \setminus Z$  (see, for example, [NR2] for an elementary proof). In particular, if the levels of the holomorphic map  $f: X \rightarrow \mathbb{P}^1$  are compact, then Stein factorization gives a surjective proper holomorphic mapping of  $X$  onto a Riemann surface.

We will say that a complete Hermitian manifold  $(X, g)$  has *bounded geometry of order  $k$*  if, for some constant  $C > 0$  and for every point  $p \in X$ , there is a biholomorphism  $\Psi$  of the unit ball  $B = B(0; 1) \subset \mathbb{C}^n$  onto a neighborhood of  $p$  in  $X$  such that  $\Psi(0) = p$  and, on  $B$ ,

$$C^{-1}g_{\mathbb{C}^n} \leq \Psi^*g \leq Cg_{\mathbb{C}^n} \quad \text{and} \quad |D^m \Psi^*g| \leq C \text{ for } m = 0, 1, 2, \dots, k.$$

For  $k = 0$ , we will simply say that  $(X, g)$  has *bounded geometry*. Gromov [Gro2] observed that, for  $f = \omega_1/\omega_2$  as above, one gets compact levels if  $X$  is a bounded geometry complete Kähler manifold and the 1-forms are in  $L^2$  and have exact real parts; thus giving an  $L^2$  version of the Castelnuovo-de Franchis theorem. He also introduced his so-called *cup product lemma*, according to which, two  $L^2$  holomorphic 1-forms  $\omega_1$  and  $\omega_2$  with exact real parts on a bounded geometry complete Kähler manifold must satisfy  $\omega_1 \wedge \omega_2 \equiv 0$ . He applied these results to the study of Kähler groups. Other versions have since been developed and applied by others in many different contexts. Other versions and applications of the Castelnuovo-de Franchis theorem (for compact and noncompact manifolds) and the cup product lemma appear in, for example, [Siu2], [CarT], [Gro1], [L], [Gro2], the work of Beauville (see [Cat]), [Sim1], [GroS], [ArBR], [JsY1], [JsY2], [Sim2], [Ar], [NR1], [ABCKT], [M], [JsZ], [NR2], [NR3], [DelG], [NR4], and [NR5]. In this paper, new approaches to the proofs of the  $L^2$  Castelnuovo-de Franchis theorem and to the cup product lemma are developed. These new approaches are simpler than previous approaches and give more general results. In particular, a version of the  $L^2$  Castelnuovo-de Franchis theorem is obtained in which only *one* of the holomorphic 1-forms need be in  $L^2$ .

**Theorem 0.1** ( $L^2$  Castelnuovo-de Franchis theorem). *Let  $(X, g)$  be a connected complete Kähler manifold with bounded geometry and let  $\omega_1$  and  $\omega_2$  be linearly independent closed holomorphic 1-forms on  $X$  such that  $\omega_1$  is in  $L^2$  and  $\omega_1 \wedge \omega_2 \equiv 0$ . Then there exist a*

surjective proper holomorphic mapping  $\Phi: X \rightarrow S$  of  $X$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_X = \mathcal{O}_S$  and holomorphic 1-forms  $\theta_1$  and  $\theta_2$  on  $S$  such that  $\omega_j = \Phi^*\theta_j$  for  $j = 1, 2$ .

The main point of the proof is that, for a suitable small open set, the holonomy induced by the holomorphic foliation associated to the holomorphic 1-forms is trivial (see Section 1). A version for a bounded geometry (of order 2) end is also obtained (Theorem 6.1).

For the cup product lemma, the main point is that one obtains different versions by considering positive forms rather than just holomorphic 1-forms; an observation which has its roots in the theory of currents and which has been applied in other contexts to obtain related results. Simple Stokes theorem arguments together with Gromov's arguments then give myriad versions of which only a few will be considered in this paper (see Sections 2 and 5). For example, there is the following version in which one of the forms is assumed to be in  $L^\infty$  instead of in  $L^2$  and the other form need not have exact real part:

**Theorem 0.2.** *Let  $\omega_1$  and  $\omega_2$  be closed holomorphic 1-forms on a connected complete Kähler manifold  $X$  such that  $\omega_1$  is bounded,  $\text{Re}(\omega_1)$  is exact, and  $\omega_2$  is in  $L^2$ . Then  $\omega_1 \wedge \omega_2 = 0$ .*

*Remark.* By the Gaffney theorem [Ga], an  $L^2$  holomorphic 1-form on a complete Kähler manifold is automatically closed, so the requirement that  $\omega_2$  be closed is superfluous.

Theorem 0.1 and Theorem 0.2 together give the following:

**Corollary 0.3.** *Let  $\omega_1$  and  $\omega_2$  be linearly independent closed holomorphic 1-forms on a connected complete Kähler manifold  $X$  with bounded geometry such that  $\omega_1$  is bounded,  $\text{Re}(\omega_1)$  is exact, and  $\omega_2$  is in  $L^2$ . Then there exist a surjective proper holomorphic mapping  $\Phi: X \rightarrow S$  of  $X$  onto a Riemann surface  $S$  and holomorphic 1-forms  $\theta_1$  and  $\theta_2$  on  $S$  such that  $\omega_j = \Phi^*\theta_j$  for  $j = 1, 2$ .*

*Remark.* Since an  $L^2$  holomorphic 1-form on a bounded geometry complete Kähler manifold is bounded, the condition that  $\omega_1$  is bounded may be replaced with the condition that  $\omega_1$  is in  $L^2$ .

The proof of Theorem 0.1 appears in Section 1 and that of Theorem 0.2 in Section 2. As an application, the results are shown in Sections 3 and 4 to give a slightly simplified proof of the main result of [NR5]. Further generalizations of the cup product lemma appear in

Section 5. Finally, a version of the  $L^2$  Castelnuovo-de Franchis theorem for an end (which is applied in [NR6]) is proved in Section 6.

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## 1. PROOF OF THE $L^2$ CASTELNUOVO-DE FRANCHIS THEOREM

Given two linearly independent closed holomorphic 1-forms  $\omega_1$  and  $\omega_2$  on a connected complex manifold  $X$  with  $\omega_1 \wedge \omega_2 \equiv 0$ , we get a nonconstant holomorphic map

$$f = \frac{\omega_1}{\omega_2}: X \rightarrow \mathbb{P}^1.$$

We have  $(f_*) \wedge \omega_1 = (f_*) \wedge \omega_2 \equiv 0$  since, on  $f^{-1}(\mathbb{C}) = f^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ ,  $df \wedge \omega_2 = d\omega_1 = 0$ . It follows that  $f$  is locally constant on the analytic set

$$Z = \{x \in X \mid (\omega_1)_x = 0 \text{ or } (\omega_2)_x = 0\}$$

(in particular,  $f(Z)$  is countable) and  $f$  is constant on each leaf of the holomorphic foliation determined by  $\omega_1$  and  $\omega_2$  in  $X \setminus Z$ . Thus  $\omega_1$  and  $\omega_2$  determine a singular holomorphic foliation in  $X$  with closed leaves given by the levels of  $f$ . Moreover, for  $j = 1, 2$ ,  $\omega_j$  is exact in a neighborhood of each level  $L$  of  $f$ . For the integral of  $\omega_j$  along any closed loop in  $L$  and, therefore, along any closed loop in a small neighborhood of  $L$ , must be zero.

The main step in the proof of Theorem 0.1 is the following:

**Lemma 1.1.** *Let  $(X, g)$  be a connected Hermitian manifold. If  $\omega_1$  and  $\omega_2$  are two linearly independent closed holomorphic 1-forms on  $X$ ,  $\omega_1$  is in  $L^2$ ,  $\omega_1 \wedge \omega_2 \equiv 0$ , and  $f = \omega_1/\omega_2: X \rightarrow \mathbb{P}^1$ , then the levels of  $f$  over almost every regular value have finite volume (that is, almost every smooth (closed) leaf of the holomorphic foliation determined by  $\omega_1$  and  $\omega_2$  has finite volume).*

*Proof.* Given a regular value  $\zeta_0 \in \mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$  of  $f$  and a point  $p \in f^{-1}(\zeta_0)$ , we may choose a relatively compact holomorphic coordinate neighborhood  $(U, z = (z_1, \dots, z_n))$  in  $X$  in which  $f|_U = z_1$ ,  $p = (\zeta_0, 0, \dots, 0)$ , and  $U = D \times \Delta^{n-1}$  where  $D$  is a disk centered at  $\zeta_0$  and  $\Delta$  is a disk centered at 0 in  $\mathbb{C}$ ; and we may choose a holomorphic function  $h$  on  $U$  with  $\omega_1|_U = dh$ .

If  $A = D \times \{0\} \subset U$  and  $\Omega$  is the union of all of those levels of  $f$  which meet  $A$ , then  $\Omega$  is a nonempty connected open subset of  $X$  containing  $U$ . For if  $\{x_\nu\}$  is a sequence in  $X$  converging to a point  $y \in \Omega$ ,  $L$  is the level containing  $y$ , and  $L_\nu$  is the level containing  $x_\nu$  for

each  $\nu$ , then, by continuity of intersections (see [Ste], [TW], and Section 4.3 of [ABCKT]), after replacing the sequence with a suitable subsequence, we get  $L_\nu \rightarrow L$ . Since  $L$  meets  $U$ , we have  $L_\nu \cap U \neq \emptyset$ , and hence  $L_\nu \cap A \neq \emptyset$ , for  $\nu \gg 0$  ( $L_\nu \cap U$  and  $L \cap U$  are slices of the form  $\{\zeta\} \times \Delta^{n-1}$ ). Thus  $x_\nu \in \Omega$  for  $\nu \gg 0$  and it follows that  $\Omega$  is open.

Since  $dh \wedge dz_1 = \omega_1 \wedge df \equiv 0$ ,  $h$  is constant in the variables  $(z_2, \dots, z_n)$  in  $U$  and we have  $h = k(z_1)$  on  $U$  for some nonconstant holomorphic function  $k$  on  $D$ . Thus, since  $f(\Omega) = f(U) = D$ , we may form the holomorphic extension  $h_0 = k(f)$  on  $\Omega$  and, since  $dh = \omega_1$  on  $U$ , we get  $dh_0 = \omega_1$  on  $\Omega$ . In particular, since  $\omega_1$  is in  $L^2$ ,  $h_0$  must have finite energy. Setting  $u = \operatorname{Re}(h_0)$  and  $v = \operatorname{Im}(h_0)$  and applying the coarea formula to the mapping  $(u, v): \Omega \rightarrow \mathbb{R}^2$ , we see that there exists a set  $S'$  of measure 0 in  $\mathbb{C}$  such that  $\operatorname{vol}(h_0^{-1}(\zeta)) < \infty$  for each  $\zeta \in \mathbb{C} \setminus S'$ . We may choose a set  $S$  of measure 0 in  $D$  which contains the set of critical values of  $f$  as well as the set  $k^{-1}(S')$ . For each point  $\zeta \in D \setminus S$ , the level  $L = f^{-1}(\zeta) \cap \Omega$  of  $f$  over  $\zeta$  meeting  $\Omega$  is a connected component of  $h_0^{-1}(k(\zeta))$  and, therefore,  $\operatorname{vol}(L) < \infty$ .

Finally, forming a countable collection  $\{U_\nu\}$  of such open sets  $U$  in  $X$  covering

$$f^{-1}(\{\text{regular values}\} \setminus \infty),$$

forming the associated measure 0 sets  $\{S_\nu\}$  in  $\mathbb{C} \subset \mathbb{P}^1$ , and letting  $S \subset \mathbb{P}^1$  be the measure 0 set given by

$$S = \bigcup_{\nu} S_\nu \cup \{\text{critical values}\} \cup \{\infty\},$$

we see that each of the levels of  $f$  over every point in  $\mathbb{P}^1 \setminus S$  has finite volume.  $\square$

Theorem 0.1 now follows from standard arguments (see [Gro2], [ArBR], and Chapter 4 of [ABCKT]) which are sketched below for the convenience of the reader.

*Proof of Theorem 0.1.* Let  $(X, g)$  be a connected complete Kähler manifold with bounded geometry and let  $\omega_1$  and  $\omega_2$  be two linearly independent closed holomorphic 1-forms such that  $\omega_1$  is in  $L^2$  and  $\omega_1 \wedge \omega_2 \equiv 0$ . We may also assume that  $n = \dim X > 1$ . The holomorphic map  $f = \omega_1/\omega_2: X \rightarrow \mathbb{P}^1$  is open, and, by Lemma 1.1, we may fix a regular value  $\zeta_0 \in f(X) \setminus \{\infty\}$  and a connected component  $L_0$  of the submanifold  $f^{-1}(\zeta_0)$  of  $X$  such that  $\operatorname{vol}(L_0) < \infty$ . Lelong's monotonicity formula (see 15.3 in [Chi]) shows that there is a constant  $c > 0$  such that each point  $p \in X$  has a neighborhood  $U_p$  such that

$\text{diam}(U_p) < 1$  and  $\text{vol}(A \cap U_p) \geq c$  for every complex analytic set  $A$  of pure dimension  $n - 1$  in  $X$  with  $p \in A$ . Therefore, since  $L_0$  has finite volume,  $L_0$  must be compact.

It follows that the set  $V = \{x \in X \mid x \text{ lies in a compact level of } f\}$  is a nonempty open set. To show that  $V$  is also closed, let  $V_0$  be a component of  $V$ , let  $\{x_j\}$  be a sequence in  $V_0$  converging to a point  $p \in \overline{V_0}$ , and, for each  $j$ , let  $L_j \subset V_0$  be the compact level of  $f$  through  $x_j$ . Stein factoring  $f|_{V_0}$ , we get a proper holomorphic mapping  $\Phi: V_0 \rightarrow S$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_{V_0} = \mathcal{O}_S$ . We may choose each  $x_j$  to lie over a regular value of  $f$  and of  $\Phi$ . Applying Stokes' theorem as in [Sto], we see that  $\text{vol}(L_j)$  is constant in  $j$  and so the above volume estimate implies that, for some  $R \gg 0$ , we have  $L_j \subset B(p; R)$  for  $j = 1, 2, 3, \dots$ . On the other hand, by [Ste] (see also [TW] or Theorem 4.23 in [ABCKT]), a subsequence of  $\{L_j\}$  converges to the level  $L$  of  $f$  through  $p$ . So we must have  $L \subset \overline{B(p; R)}$  and hence  $L$  is compact. Thus  $p \in \overline{V_0} \cap V$  and, therefore,  $p \in V_0$ . It follows that  $V = V_0 = X$ . Thus every level of  $f$  is compact and we get our proper holomorphic mapping  $\Phi: X \rightarrow S$ .

Finally, we recall that, for each  $j = 1, 2$ ,  $\omega_j$  is exact on a neighborhood of each level of  $f$ ; that is, on a neighborhood of each fiber of  $\Phi$ . Thus, for each point  $s \in S$ , we have a connected neighborhood  $D$  of  $s$  in  $S$  and a holomorphic function  $h_j$  on  $U = \Phi^{-1}(D)$  such that  $\omega_j = dh_j$  on  $U$ . The function  $h_j$  descends to a unique holomorphic function  $k_j$  on  $D$  with  $\Phi^*k_j = h_j$ . Thus we get a unique well-defined holomorphic 1-form  $\theta_j$  on  $S$  with  $\Phi^*\theta_j = \omega_j$  by setting  $\theta_j|_D = dk_j$  on each such neighborhood  $D$ .  $\square$

The following easy consequence is a more convenient form for some applications:

**Corollary 1.2.** *Let  $(X, g)$  be a connected complete Kähler manifold with bounded geometry and let  $\rho_1$  and  $\rho_2$  be two real-valued pluriharmonic functions on  $X$  such that  $d\rho_1$  and  $d\rho_2$  are linearly independent,  $\rho_1$  has finite energy, and  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$ . Then there exist a surjective proper holomorphic mapping  $\Phi: X \rightarrow S$  of  $X$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_X = \mathcal{O}_S$  and real-valued pluriharmonic functions  $\alpha_1$  and  $\alpha_2$  on  $S$  such that  $\rho_j = \Phi^*\alpha_j$  for  $j = 1, 2$ .*

*In particular, if there exists a nonconstant holomorphic function with finite energy on  $X$ , then there exists a surjective proper holomorphic mapping  $\Phi: X \rightarrow S$  of  $X$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_X = \mathcal{O}_S$ .*

*Remark.* Two real-valued pluriharmonic functions  $u$  and  $v$  on a connected complex manifold have linearly dependent differentials (i.e. the functions  $u$ ,  $v$ , and 1 are linearly dependent) if and only if  $du \wedge dv \equiv 0$ .

*Proof of Corollary 1.2.* If  $\partial\rho_1$  and  $\partial\rho_2$  are linearly independent, then we may apply Theorem 0.1 to this pair of holomorphic 1-forms. If not, then there exist constants  $\zeta_1, \zeta_2 \in \mathbb{C} \setminus \{0\}$  such that the function  $h = \zeta_1\rho_1 + \zeta_2\rho_2: X \rightarrow \mathbb{C}$  is a nonconstant holomorphic function with finite energy. The closed holomorphic 1-forms  $\omega_1 \equiv dh$  and  $\omega_2 \equiv h dh = 2^{-1}d(h^2)$  are then linearly independent and  $\omega_1$  is in  $L^2$ , so we may again apply Theorem 0.1. In either case, we get a proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $X$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_X = \mathcal{O}_S$  and the pluriharmonic functions  $\rho_1$  and  $\rho_2$  descend to pluriharmonic functions  $\alpha_1$  and  $\alpha_2$ , respectively, on  $S$ .  $\square$

**Definition 1.3.** For  $S \subset X$  and  $k$  a positive integer, we will say that a Hermitian manifold  $(X, g)$  has *bounded geometry of order  $k$  along  $S$*  if, for some constant  $C > 0$  and for every point  $p \in S$ , there is a biholomorphism  $\Psi$  of the unit ball  $B = B(0; 1) \subset \mathbb{C}^n$  onto a neighborhood of  $p$  in  $X$  such that  $\Psi(0) = p$  and such that, on  $B$ ,

$$C^{-1}g_{\mathbb{C}^n} \leq \Psi^*g \leq Cg_{\mathbb{C}^n} \quad \text{and} \quad |D^m \Psi^*g| \leq C \text{ for } m = 0, 1, 2, \dots, k.$$

Slight modifications of the proofs of Theorem 0.1 and Corollary 1.2 give the following useful generalizations:

**Theorem 1.4.** *Let  $\Omega$  be a nonempty domain in a connected complete Hermitian manifold  $(X, g)$  and let  $\omega_1$  and  $\omega_2$  be linearly independent closed holomorphic 1-forms on  $\Omega$  such that  $X$  has bounded geometry along  $\Omega$ ,  $g|_{\Omega}$  is Kähler,  $\omega_1$  is in  $L^2$ ,  $\omega_1 \wedge \omega_2 \equiv 0$  on  $\Omega$ , and the levels of the associated holomorphic mapping  $f = (\omega_1/\omega_2): \Omega \rightarrow \mathbb{P}^1$  are closed relative to  $X$ . Then there exist a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $\Omega$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_{\Omega} = \mathcal{O}_S$  and holomorphic 1-forms  $\theta_1$  and  $\theta_2$  on  $S$  such that  $\omega_j = \Phi^*\theta_j$  for  $j = 1, 2$ .*

**Corollary 1.5.** *Let  $\Omega$  be a nonempty domain in a connected complete Hermitian manifold  $(X, g)$  and let  $\rho_1$  and  $\rho_2$  be two real-valued pluriharmonic functions on  $\Omega$  such that  $d\rho_1$  and  $d\rho_2$  are linearly independent,  $X$  has bounded geometry along  $\Omega$ ,  $g|_{\Omega}$  is Kähler,  $\rho_1$  has finite energy,  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$  on  $\Omega$ , and the closure (relative to  $X$ ) of each leaf of the (singular) holomorphic foliation determined by  $\partial\rho_1$  (and  $\partial\rho_2$ ) is contained in  $\Omega$ . Then there exist*

a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $\Omega$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$  and real-valued pluriharmonic functions  $\alpha_1$  and  $\alpha_2$  on  $S$  such that  $\rho_j = \Phi^*\alpha_j$  for  $j = 1, 2$ .

In particular, if there exists a nonconstant holomorphic function with finite energy on  $\Omega$  whose levels are closed relative to  $X$ , then there exists a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $\Omega$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$ .

## 2. PROOF OF THE CUP PRODUCT LEMMA

Throughout this section  $(X, g)$  will denote a connected complete Hermitian manifold of dimension  $n$  with associated real  $(1, 1)$ -form  $\eta$ . As in [Ga], fixing a point  $p \in X$  and setting

$$\tau(s) = \begin{cases} 1 & \text{if } s \leq 1 \\ 2 - s & \text{if } 1 < s < 2 \\ 0 & \text{if } 2 \leq s \end{cases}$$

and

$$\tau_r(x) = \tau\left(\frac{\text{dist}(p, x)}{r}\right)$$

for each point  $x \in X$  and each number  $r > 0$ , we get a collection of nonnegative Lipschitz continuous functions  $\{\tau_r\}_{r>0}$  such that, for each  $r > 0$ , we have  $0 \leq \tau_r \leq 1$  on  $X$ ,  $\tau_r \equiv 1$  on  $B(p; r)$ ,  $\tau_r \equiv 0$  on  $X \setminus B(p; 2r)$ , and  $|d\tau_r|_g \leq 1/r$ . Finally, for each  $R > 0$ ,  $\mathcal{M}_R$  will denote the operator given by

$$\mathcal{M}_R(\varphi)(x) = \begin{cases} \varphi(x) & \text{if } |\varphi(x)| \leq R \\ R & \text{if } \varphi(x) > R \\ -R & \text{if } \varphi(x) < -R \end{cases}$$

for every (extended) real-valued function  $\varphi$ .

*Proof of Theorem 0.2.* Clearly, we may assume that  $n = \dim X > 1$ . Assuming  $X$  is Kähler, let  $\omega_1$  and  $\omega_2$  be closed holomorphic 1-forms on  $X$  such that  $\omega_1$  is bounded,  $\text{Re}(\omega_1)$  is exact, and  $\omega_2$  is in  $L^2$ . In particular, we may fix a real-valued pluriharmonic function  $\rho$  on  $X$  such that  $\text{Re}(\omega_1) = d\rho$ . Setting  $d^c = -\sqrt{-1}(\partial - \bar{\partial})$ , we get

$$0 \leq \sqrt{-1}\omega_1 \wedge \bar{\omega}_1 = dd^c(\rho^2) = 2d(\rho d^c \rho),$$

and hence  $\gamma = d\theta$ , where  $\gamma$  is the nonnegative form of type  $(n, n)$  given by

$$\gamma \equiv (\sqrt{-1}\omega_1 \wedge \bar{\omega}_1) \wedge (\sqrt{-1}\omega_2 \wedge \bar{\omega}_2) \wedge \eta^{n-2}$$



and

$$\theta \equiv 2\rho(d^c\rho) \wedge (\sqrt{-1}\omega_2 \wedge \overline{\omega_2}) \wedge \eta^{n-2}.$$

For every  $R > 0$ , let  $\gamma_R$  be the product of  $\gamma$  and the characteristic function of

$$\{x \in X \mid |\rho(x)| \leq R\},$$

let  $\rho_R = \mathcal{M}_R(\rho)$ , and let  $\theta_R$  be the  $L^1$  Lipschitz continuous form given by

$$\theta_R \equiv 2\rho_R(d^c\rho) \wedge (\sqrt{-1}\omega_2 \wedge \overline{\omega_2}) \wedge \eta^{n-2}.$$

Then, for almost every  $R > 0$ ,  $\gamma_R$  is equal almost everywhere to  $d\theta_R$ ; in fact,  $\gamma_R = d\theta_R$  on  $X \setminus \rho^{-1}(\{\pm R\})$ . For each such fixed  $R > 0$  and each  $r > 0$ , Stokes' theorem gives.

$$\int_X \tau_r \gamma_R = - \int_X d\tau_r \wedge \theta_R.$$

Letting  $r \rightarrow \infty$  and applying the dominated convergence theorem on the right-hand side, we get

$$\int_X \gamma_R = 0.$$

We have  $\gamma_R \geq 0$ , and, therefore,  $\gamma_R = 0$ , on  $X \setminus \rho^{-1}(\{\pm R\})$ . Letting  $R \rightarrow \infty$ , we get  $\gamma \equiv 0$  on  $X$  and it follows that  $\omega_1 \wedge \omega_2 \equiv 0$ .  $\square$

Similar arguments yield generalizations; several examples of which will be considered in Section 5. For now, we consider two slight generalizations of Lemma 2.7 of [NR5] which will also allow us to give a simplified proof of the main result of [NR5] (see Sections 3 and 4). The proof given below is also simpler than the proof of Lemma 2.7 of [NR5] given in that paper.

**Theorem 2.1.** *Let  $\omega_1$  and  $\omega_2$  be two closed holomorphic 1-forms on a domain  $Y \subset X$  such that  $\operatorname{Re}(\omega_1) = d\rho_1$  for some real-valued pluriharmonic function  $\rho_1$  on  $Y$ . Assume that, for some constant  $a$  with  $\inf \rho_1 < a < \sup \rho_1$  and some component  $\Omega$  of  $\{x \in Y \mid a < \rho_1(x)\}$ , we have the following:*

- (i)  $\overline{\Omega} \subset Y$ ;
- (ii) The metric  $g \upharpoonright_{\Omega}$  is Kähler;
- (iii) The form  $\omega_1 \upharpoonright_{\Omega}$  is bounded; and
- (iv)  $\int_{\Omega} |\omega_2|_g^2 dV_g < \infty$ .

Then  $\omega_1 \wedge \omega_2 \equiv 0$  on  $Y$ . Furthermore, if  $\omega_1$  and  $\omega_2$  are linearly independent and  $(X, g)$  has bounded geometry along  $\Omega$ , then there exist a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $\Omega$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$  and holomorphic 1-forms  $\theta_1$  and  $\theta_2$  on  $S$  such that  $\omega_j|_\Omega = \Phi^*\theta_j$  for  $j = 1, 2$ .

*Proof.* Clearly, we may assume that  $n = \dim X > 1$ . Let  $\gamma$  be the nonnegative form of type  $(n, n)$  on  $Y$  given by

$$\gamma \equiv (\sqrt{-1}\omega_1 \wedge \overline{\omega_1}) \wedge (\sqrt{-1}\omega_2 \wedge \overline{\omega_2}) \wedge \eta^{n-2}.$$

Fixing a regular value  $b$  for  $\rho_1$  with  $a < b < \sup_\Omega \rho_1$ , setting  $\Omega_b = \{x \in \Omega \mid b < \rho_1(x)\} \neq \emptyset$ , and setting

$$\theta \equiv 2(\rho_1 - b)(d^c \rho_1) \wedge (\sqrt{-1}\omega_2 \wedge \overline{\omega_2}) \wedge \eta^{n-2},$$

we get  $\gamma = d\theta$  on  $\Omega$ . For every  $R > 0$ , let  $\gamma_R$  be the product of  $\gamma$  and the characteristic function of

$$\{x \in Y \mid |\rho_1(x) - b| \leq R\},$$

let  $\alpha_R = \mathcal{M}_R(\rho_1 - b)$ , and let  $\theta_R$  be the  $L^1$  Lipschitz continuous form on  $\Omega$  given by

$$\theta_R \equiv 2\alpha_R(d^c \rho_1) \wedge (\sqrt{-1}\omega_2 \wedge \overline{\omega_2}) \wedge \eta^{n-2}.$$

Then, for almost every  $R > 0$ ,  $\gamma_R$  is equal almost everywhere to  $d\theta_R$  in  $\Omega$ ; in fact,  $\gamma_R = d\theta_R$  on  $\Omega \setminus \rho_1^{-1}(\{b \pm R\})$ . For each such fixed  $R > 0$  and each  $r > 0$ , Stokes' theorem gives

$$\int_{\Omega_b} \tau_r \gamma_R = - \int_{\Omega_b} d\tau_r \wedge \theta_R;$$

since  $\alpha_R \equiv 0$  on  $\partial\Omega_b$ . Letting  $r \rightarrow \infty$  and applying the dominated convergence theorem on the right-hand side, we get

$$\int_{\Omega_b} \gamma_R = 0.$$

We have  $\gamma_R \geq 0$ , and, therefore,  $\gamma_R = 0$ , on  $\Omega_b \setminus \rho_1^{-1}(\{b \pm R\})$ . Letting  $R \rightarrow \infty$ , we get  $\gamma \equiv 0$  on  $\Omega_b$  and it follows that  $\omega_1 \wedge \omega_2 \equiv 0$  on  $Y$ .

Assume now that  $\omega_1$  and  $\omega_2$  are linearly independent and  $(X, g)$  has bounded geometry along  $\Omega$ . Since  $\rho_1$  is constant on the levels of the holomorphic map  $f = \omega_1/\omega_2$ , those levels which meet  $\Omega$  are contained in  $\Omega$ . Thus Theorem 1.4 gives the desired proper holomorphic mapping to a Riemann surface.  $\square$

Applying the above theorem together with Corollary 1.5, we get the following:

**Corollary 2.2.** *Let  $\rho_1$  and  $\rho_2$  be two real-valued pluriharmonic functions on a domain  $Y \subset X$ . Assume that, for some constant  $a$  with  $\inf \rho_1 < a < \sup \rho_1$  and some component  $\Omega$  of  $\{x \in Y \mid a < \rho_1(x)\}$ , we have the following:*

- (i)  $\overline{\Omega} \subset Y$ ,
- (ii) *The metric  $g \upharpoonright_{\Omega}$  is Kähler,*
- (iii) *The form  $d\rho_1 \upharpoonright_{\Omega}$  is bounded, and*
- (iv)  $\int_{\Omega} |d\rho_2|_g^2 dV_g < \infty$ .

*Then  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$  on  $Y$ . Furthermore, if  $d\rho_1$  and  $d\rho_2$  are linearly independent and  $(X, g)$  has bounded geometry along  $\Omega$ , then there exist a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $\Omega$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_{\Omega} = \mathcal{O}_S$  and pluriharmonic functions  $\alpha_1$  and  $\alpha_2$  on  $S$  such that  $\rho_j \upharpoonright_{\Omega} = \Phi^*\alpha_j$  for  $j = 1, 2$ .*

### 3. AN APPLICATION TO FILTERED ENDS OF KÄHLER MANIFOLDS

Let  $X$  be a connected complete Kähler manifold. According to [Gro1], [L], [Gro2], and Theorem 3.4 of [NR1], if  $X$  has at least 3 ends, and either  $X$  has bounded geometry of order 2 or  $X$  is weakly 1-complete or  $X$  admits a positive symmetric Green's function which vanishes at infinity, then  $X$  maps properly and holomorphically onto a Riemann surface. The ends condition was weakened in [DelG] and [NR5] to the condition that  $X$  have at least 3 *filtered ends* relative to the universal covering. The techniques and results described in the previous sections allow one to simplify the proof of the main result of [NR5] (see Theorem 0.1 and Theorem 3.1 of [NR5]) in the following sense. The proof given in [NR5] relied heavily on a weak version of Theorem 2.1 in which both of the holomorphic 1-forms are assumed to be in  $L^2$  on the domain  $\Omega$  and to have exact real parts (see Lemma 2.7 of [NR5]). The proof of Theorem 2.1 given in Section 2 is simpler than that of the weak version given in [NR5]. Moreover, Theorem 2.1, being stronger, allows one to eliminate some of the technical arguments used in [NR5]. In fact, one can avoid any direct use of the general theory of massive sets due to Grigor'yan [Gri]; a central technique employed in [NR5]. In this section, we recall the required definitions and preliminary facts. The new proof appears in Section 4.

**Definition 3.1.** Let  $M$  be a connected manifold.

- (a) By an *end* of  $M$ , we will mean either a component  $E$  of  $M \setminus K$  with noncompact closure, where  $K$  is a given compact subset of  $M$ , or an element of

$$\lim_{\leftarrow} \pi_0(M \setminus K),$$

where the limit is taken as  $K$  ranges over the compact subsets of  $M$  (or the compact subsets of  $M$  whose complement  $M \setminus K$  has no relatively compact components). The number of ends of  $M$  will be denoted by  $e(M)$ . For a compact set  $K$  such that  $M \setminus K$  has no relatively compact components, we will call

$$M \setminus K = E_1 \cup \cdots \cup E_m,$$

where  $E_1, \dots, E_m$  are the distinct components of  $M \setminus K$ , an *ends decomposition* for  $M$ .

- (b) (Following Geoghegan [Ge]) For  $\Upsilon : \widetilde{M} \rightarrow M$  the universal covering of  $M$ , elements of the set

$$\lim_{\leftarrow} \pi_0[\Upsilon^{-1}(M \setminus K)],$$

where the limit is taken as  $K$  ranges over the compact subsets of  $M$  (or the compact subsets of  $M$  whose complement  $M \setminus K$  has no relatively compact components) will be called *filtered ends*. The number of filtered ends of  $M$  will be denoted by  $\tilde{e}(M)$ .

Clearly,  $\tilde{e}(M) \geq e(M)$ . In fact, for  $k \in \mathbb{N}$ , we have  $\tilde{e}(M) \geq k$  if and only if there exists an ends decomposition  $M \setminus K = E_1 \cup \cdots \cup E_m$  for  $M$  such that, for  $\Gamma_j = \text{im} [\pi_1(E_j) \rightarrow \pi_1(M)]$  for  $j = 1, \dots, m$ , we have

$$\sum_{j=1}^m [\pi_1(M) : \Gamma_j] \geq k.$$

Moreover, if  $\widehat{M} \rightarrow M$  is a connected covering space, then  $\tilde{e}(\widehat{M}) \leq \tilde{e}(M)$  with equality if the covering is finite.

**Definition 3.2.** We will say that a complex manifold  $X$  is *weakly 1-complete along* a subset  $S$  if there exists a continuous plurisubharmonic function  $\varphi$  on  $X$  such that

$$\{x \in S \mid \varphi(x) < a\} \Subset X \quad \forall a \in \mathbb{R}.$$

**Definition 3.3.** We will call an end  $E$  of a connected noncompact complete Hermitian manifold  $(X, g)$  *special* if  $E$  is of at least one of the following types:

- (BG)  $(X, g)$  has bounded geometry of order 2 along  $E$ ;

- (W)  $X$  is weakly 1-complete along  $E$ ;
- (RH)  $E$  is a hyperbolic end and the Green's function vanishes at infinity along  $E$ ; or
- (SP)  $E$  is a parabolic end, the Ricci curvature of  $g$  is bounded below on  $E$ , and there exist positive constants  $R$  and  $\delta$  such that

$$\text{vol}(B(p; R)) > \delta \quad \forall p \in E.$$

An ends decomposition for  $X$  in which each of the ends is special will be called a *special ends decomposition*.

*Remarks.* 1. (BG) stands for “bounded geometry,” (W) for “weakly 1-complete,” (RH) for “regular hyperbolic,” and (SP) for “special parabolic.”

- 2. A parabolic end of type (BG) is also of type (SP).
- 3. If  $E$  and  $E'$  are ends with  $E' \subset E$  and  $E$  is special, then  $E'$  is special.
- 4. We recall that an end  $E$  of a connected Riemannian manifold  $(M, g)$  is hyperbolic if and only if there exists a bounded nonnegative continuous subharmonic function  $\alpha$  on  $M$  such that  $\alpha \equiv 0$  on  $M \setminus E$  and  $\sup_E \alpha > 0$ . Such a function  $\alpha$  is called an *admissible subharmonic function* for  $E$  in  $M$ . The end  $E$  is special of type (RH) if and only if we may choose  $\alpha$  so that  $\alpha \rightarrow \sup \alpha$  at infinity in  $\overline{E}$ . As in the work of Grigor'yan [Gri], any open set  $E$  (whether or not it's an end) is called *massive* if there exists an admissible subharmonic function for  $E$ . General massive sets are applied in [NR5], but the results of Section 2 will allow us to restrict our attention to hyperbolic ends.

Special ends in a complete Kähler manifold allow one to produce pluriharmonic functions and, in some cases, holomorphic functions. In particular, one gets the following:

**Theorem 3.4** ([Gro1], [L], [Gro2], and Theorem 3.4 of [NR1]). *If  $(X, g)$  is a connected complete Kähler manifold which admits a special ends decomposition and  $e(X) \geq 3$ , then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

The main result of [NR5] is the following generalization (see Theorem 3.1 of [NR5]):

**Theorem 3.5.** *If  $(X, g)$  is a connected complete Kähler manifold which admits a special ends decomposition and  $\tilde{e}(X) \geq 3$ , then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

The goal of this section and Section 4 is to describe a simpler proof of the above fact. We will produce independent pluriharmonic functions by applying Theorem 2.6 of [NR1], which is contained implicitly in the work of Sario, Nakai, and their collaborators [Na1],[Na2], [SaNa], [SaNo], [RoS] and the work of Sullivan [Sul] (see also [L] and [LT]). This fact is also applied in [NR5] along with the more general theory of massive sets [Gri], but we will not need general massive sets in this paper. In fact, we will only need the following weak version of Theorem 2.6 of [NR1]:

**Theorem 3.6.** *Let  $(X, g)$  be a connected complete Kähler manifold with an ends decomposition  $X \setminus K = E_1 \cup \dots \cup E_m$  such that  $m > 1$  and such that, for each  $j = 1, \dots, m$ ,  $E_j$  is a hyperbolic end or a special end of type (SP). Then there exists a pluriharmonic function  $\rho: X \rightarrow \mathbb{R}$  such that, for each  $j = 1, \dots, m$ , we have the following:*

- (i) *If  $E_j$  is a hyperbolic end, then  $0 < \rho|_{E_j} < 1$  and  $\rho|_{E_j}$  has finite energy;*
- (ii) *If  $E_1$  is a hyperbolic end (a special end of type (RH)), then*

$$\limsup_{x \rightarrow \infty} \rho|_{\overline{E_1}}(x) = 1 \quad (\text{respectively, } \lim_{x \rightarrow \infty} \rho|_{\overline{E_1}}(x) = 1);$$

and

- (iii) *If  $E_1$  is a special end of type (SP), then*

$$\lim_{x \rightarrow \infty} \rho|_{\overline{E_1}}(x) = \infty.$$

*Remark.* Theorem 2.6 of [NR1] is stated for dimension  $n > 1$ , but it actually holds in arbitrary dimension. On the other hand, we will only need Theorem 3.6 for  $n > 1$ .

*Proof of Theorem 3.6.* Applying Theorem 2.6 of [NR1], we get a nonconstant pluriharmonic function  $\alpha: X \rightarrow \mathbb{R}$  such that, for each  $j = 1, \dots, m$ , we have the following:

(3.6.1) If  $E_j$  is a hyperbolic end, then  $\alpha|_{E_j}$  is bounded with finite energy and

$$\liminf_{x \rightarrow \infty} \alpha|_{\overline{E_j}}(x) \begin{cases} = 0 & \text{if } j = 1 \\ > 0 & \text{if } j > 1 \end{cases}$$

(3.6.2) If  $E_j$  is a special end of type (RH), then

$$\lim_{x \rightarrow \infty} \alpha|_{\overline{E_j}}(x) = \begin{cases} 0 & \text{if } j = 1 \\ 1 & \text{if } j > 1 \end{cases}$$

(3.6.3) If  $E_j$  is a special end of type (SP), then

$$\lim_{x \rightarrow \infty} \alpha|_{\overline{E_j}}(x) = \begin{cases} \infty & \text{if } j = 1 \\ \infty & \text{if } j > 1 \text{ and } X \text{ is hyperbolic (i.e. } E_i \text{ is hyperbolic for some } i) \\ -\infty & \text{if } j > 1 \text{ and } X \text{ is parabolic (i.e. } E_1, \dots, E_m \text{ are parabolic)} \end{cases}$$

Let  $H$  be the union of all of those ends  $E_j$  which are hyperbolic and fix  $s \in \mathbb{R}$  with  $\alpha < s$  on  $H$ . If  $H \neq \emptyset$ , then the maximum principle implies that  $\alpha > 0$  on  $X$ . Thus the function

$$\rho \equiv \begin{cases} 1 - (\alpha/s) & \text{if } H \neq \emptyset \text{ and } E_1 \text{ is hyperbolic} \\ \alpha/s & \text{if } H \neq \emptyset \text{ and } E_1 \text{ is parabolic} \\ \alpha & \text{if } H = \emptyset \end{cases}$$

has the required properties.  $\square$

The following lemma may be viewed as a consequence of Theorem 3.4 (see, for example, the proof of Theorem 4.6 of [NR1]):

**Lemma 3.7.** *Let  $(X, g)$  be a connected complete Kähler manifold which is compact or which admits a special ends decomposition. If some nonempty open subset of  $X$  admits a surjective proper holomorphic mapping onto a Riemann surface, then  $X$  admits a surjective proper holomorphic mapping onto a Riemann surface.*

The following easy observation will enable us to produce pluriharmonic functions by passing to a covering.

**Lemma 3.8.** *Let  $(X, g)$  be a connected complete Kähler manifold, let  $\Upsilon: \widehat{X} \rightarrow X$  be a connected covering space, and let  $\hat{g} = \Upsilon^*g$ .*

- (a) *If  $E_1$  is a hyperbolic end of  $X$ , then any end  $F$  of  $\widehat{X}$  containing a component  $E$  of  $\Upsilon^{-1}(E_1)$  is a hyperbolic end.*
- (b) *If  $E_1$  is a special end with smooth boundary and  $E$  is a component of  $\Upsilon^{-1}(E_1)$  for which the restriction  $E \rightarrow E_1$  is a finite covering, then  $E$  is a special end of the same type.*
- (c) *If  $X \setminus K = E_1 \cup \dots \cup E_m$  is an ends decomposition into hyperbolic ends with smooth boundary and  $E$  is a component of  $\Upsilon^{-1}(E_1)$  for which the restriction  $E \rightarrow E_1$  is a finite covering, then every component of  $\widehat{X} \setminus \partial E$  with noncompact closure is a hyperbolic end of  $\widehat{X}$ .*

*Proof.* Let  $X \setminus K = E_1 \cup \cdots \cup E_m$  be an ends decomposition, let  $\widehat{E}_j = \Upsilon^{-1}(E_j)$  for  $j = 1, \dots, m$ , and let  $E$  be a component of  $\widehat{E}_1$ . If  $E_1$  is a hyperbolic end,  $\alpha$  is an admissible subharmonic function for  $E_1$ , and  $F$  is an end of  $\widehat{X}$  containing  $E$ , then we have the admissible subharmonic function

$$\beta \equiv \begin{cases} \alpha \circ \Upsilon & \text{on } E \\ 0 & \text{on } \widehat{X} \setminus E \end{cases}$$

for  $F$ . Thus (a) is proved.

If  $E_1$  is a smooth domain and the restriction  $E \rightarrow E_1$  is a finite covering, then  $E$  is an end of  $\widehat{X}$  and there exist neighborhoods  $V$  and  $V_1$  of  $\overline{E}$  and  $\overline{E}_1$ , respectively, such that  $V \cap \Upsilon^{-1}(E_1) = E$  and  $V \rightarrow V_1$  is also a finite covering space. Clearly, if  $E_1$  is special of type (BG), (W), or (RH), then  $E$  is special of the same type. If  $E$  is a hyperbolic end with an admissible subharmonic function  $\alpha$ , then the function

$$\alpha_1(x) \equiv \begin{cases} \sum_{y \in \Upsilon^{-1}(x) \cap E} \alpha(y) & \text{if } x \in E_1 \\ 0 & \text{if } x \in X \setminus E_1 \end{cases}$$

is an admissible subharmonic function for  $E_1$ . It now follows easily that, if  $E_1$  is special of type (SP), then  $E$  must also be special of type (SP). Thus (b) is proved.

Finally, suppose that  $E_j$  is a hyperbolic end and a smooth domain for each  $j = 1, \dots, m$  and that the restriction  $E \rightarrow E_1$  is a finite covering. In particular, forming  $V \rightarrow V_1$  as above, we see that  $\partial E$  is compact and every component  $F$  of  $\widehat{X} \setminus \partial E$  with noncompact closure is an end. Furthermore,  $F$  must meet  $\widehat{E}_j$  for some  $j$ . For if not, then  $F$  must be a connected component of  $\Upsilon^{-1}(X \setminus \overline{E}_1)$  contained in  $\widehat{K} = \Upsilon^{-1}(K)$ . Thus we have the connected covering space  $\overline{F} \rightarrow \overline{F}_1$  and the covering space  $\partial F \rightarrow \partial F_1$  for some component  $F_1$  of  $X \setminus \overline{E}_1$  contained in  $K$  (here, we have used smoothness). Since  $\partial F \subset \partial E$  and  $\partial E \rightarrow \partial E_1$  is a finite covering space (of manifolds), we see that  $\overline{F} \rightarrow \overline{F}_1 \subset K$  is a finite cover, which contradicts the noncompactness of  $\overline{F}$ . Thus  $F$  must meet and, therefore, contain,  $\widehat{E}_j$  for some  $j$  and (a) then implies that  $F$  is a hyperbolic end. Thus (c) is proved.  $\square$

#### 4. PROOF OF THE FILTERED ENDS RESULT

The first step in the new proof of Theorem 3.5 is to reduce to the case in which all of the ends of the manifold are hyperbolic special ends of type (BG). Toward this goal, we first recall the following two facts:



**Lemma 4.1** (See Lemma 3.2 of [NR5]). *Let  $M$  be a connected noncompact  $C^\infty$  manifold and let  $k \in \mathbb{N}$ .*

- (a) *Given an end  $E$  in  $M$  with  $[\pi_1(M) : \text{im} [\pi_1(E) \rightarrow \pi_1(M)]] \geq k$ , there exists a compact set  $D \subset M$  such that, if  $\Omega$  is a domain containing  $D$ , then  $\Omega \cap E$  is an end of  $\Omega$  and, for any end  $F$  of  $\Omega$  contained in  $E$ , we have  $[\pi_1(\Omega) : \text{im} [\pi_1(F) \rightarrow \pi_1(\Omega)]] \geq k$ .*
- (b) *If  $\tilde{e}(M) \geq k$ , then there exists a compact set  $D \subset M$  such that, for every domain  $\Omega$  containing  $D$ , we have  $\tilde{e}(\Omega) \geq k$ .*

**Lemma 4.2** (See Lemma 3.3 of [NR5]). *Let  $(X, g)$  be a connected complete Kähler manifold, let  $E$  be a special end of type (W) in  $X$ , let  $k, l \in \mathbb{N}$  with*

$$\tilde{e}(X) \geq k \quad \text{and} \quad [\pi_1(X) : \text{im} [\pi_1(E) \rightarrow \pi_1(X)]] \geq l,$$

*and let  $D$  be a compact subset of  $X$ . Then there exists a domain  $X'$  in  $X$ , a complete Kähler metric  $g'$  on  $X'$ , a compact set  $K \subset X'$ , and disjoint domains  $E_0, \dots, E_m$  such that*

- (i)  *$(X \setminus E) \cup D \subset E_0$ ,  $X' \setminus K = E_0 \cup E_1 \cup E_2 \cup \dots \cup E_m$ , and  $E_0 \cap E \Subset X'$ ;*
- (ii) *On  $E_0$ ,  $g' = g$ ;*
- (iii) *For each  $j = 1, \dots, m$ ,  $E_j$  is a special end of type (RH) and (W) satisfying  $[\pi_1(X') : \text{im} [\pi_1(E_j) \rightarrow \pi_1(X')]] \geq l$ ; and*
- (iv)  *$\tilde{e}(X') \geq k$ .*

The above lemmas allow us to replace special ends of type (W) with special ends of type (RH). The following lemma will allow us to replace special ends of type (RH) with special ends of type (BG) (under the right conditions).

**Lemma 4.3.** *Let  $(X, g)$  be a connected complete Kähler manifold, let  $E$  be an end of  $X$ , let  $k, l \in \mathbb{N}$  with*

$$\tilde{e}(X) \geq k \quad \text{and} \quad [\pi_1(X) : \text{im} [\pi_1(E) \rightarrow \pi_1(X)]] \geq l,$$

*and let  $D$  be a compact subset of  $X$ . Assume that, for some  $R \in (0, \infty]$ , there exists a continuous function  $\rho: \overline{E} \rightarrow (0, R)$  such that  $\rho$  is pluriharmonic on  $E$  and*

$$\lim_{x \rightarrow \infty} \rho|_{\overline{E}}(x) = R$$

(in particular,  $E$  is a special end of type  $(W)$ ). Then there exists a domain  $X'$  in  $X$ , a complete Kähler metric  $g'$  on  $X'$ , a compact set  $K \subset X'$ , and disjoint domains  $E_0, \dots, E_m$  such that

- (i)  $(X \setminus E) \cup D \subset E_0$ ,  $X' \setminus K = E_0 \cup E_1 \cup E_2 \cup \dots \cup E_m$ , and  $E_0 \cap E \Subset X'$ ;
- (ii) We have  $g' \geq g$  on  $X'$  and  $g' = g$  on  $E_0$ ;
- (iii) For each  $j = 1, \dots, m$ ,  $E_j$  is a special end of type  $(BG)$ ,  $(RH)$ , and  $(W)$  for  $(X', g')$  satisfying  $[\pi_1(X') : \text{im} [\pi_1(E_j) \rightarrow \pi_1(X')]] \geq l$ ; and
- (iv)  $\tilde{e}(X') \geq k$ .

*Proof.* By Lemma 4.1, we may assume without loss of generality that  $D$  is nonempty and connected;  $\partial E \subset D$ ; and, if  $\Omega$  is any domain in  $X$  containing  $D$ , then  $\tilde{e}(\Omega) \geq k$ ,  $\Omega \cap E$  is an end of  $\Omega$ , and, for any end  $F$  of  $\Omega$  contained in  $E$ , we have  $[\pi_1(\Omega) : \text{im} [\pi_1(F) \rightarrow \pi_1(\Omega)]] \geq l$ . Fixing positive constants  $a, b$ , and  $c$  with  $\max_{D \cap \overline{E}} \rho < a < b < c < R$  and a  $C^\infty$  function  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  such that  $\chi' \geq 0$  and  $\chi'' \geq 0$  on  $\mathbb{R}$ ,  $\chi(t) = 0$  for  $t \leq a$ , and  $\chi(t) = t - b$  for  $t \geq c$ , we get a  $C^\infty$  plurisubharmonic function

$$\varphi \equiv \begin{cases} \chi(\rho) & \text{on } E \\ 0 & \text{on } X \setminus E \end{cases}$$

such that  $0 \leq \varphi < R - b$ ,  $\varphi \equiv 0$  on a neighborhood of  $(X \setminus E) \cup D$ , and  $\varphi = \rho - b$  on the complement in  $E$  of the compact set  $\{x \in \overline{E} \mid \varphi(x) \leq c - b\}$ . Finally, we may fix a regular value  $r$  for  $\varphi$  with  $c - b < r < R - b$ .

On the component  $X'$  of  $\{x \in E \mid \varphi(x) < r\} \cup (X \setminus E)$  containing the connected set  $(X \setminus E) \cup D$ , we may form the complete Kähler metric

$$g' \equiv g + \mathcal{L}(-\log(r - \varphi)).$$

We have  $X' \cap E \Subset X$  since  $\varphi \rightarrow R - b$  at infinity in  $\overline{E}$ . We also have  $g' = g$  on the interior  $V$  of  $\{x \in X' \mid \varphi(x) = 0\}$  and, since  $\varphi \rightarrow r$  at  $\partial X'$ , the closure of the component  $E_0$  of  $V$  containing  $(X \setminus E) \cup D$  is contained in  $X'$  and the set

$$K \equiv X' \setminus [E_0 \cup \{x \in X' \mid \varphi(x) > 0\}]$$

is compact. Furthermore, by the maximum principle, each of the components  $E_1, \dots, E_m$  of the nonempty set  $\{x \in X' \mid \varphi(x) > 0\}$  is not relatively compact in  $X'$  and is, therefore, a special end of type  $(RH)$  and  $(W)$  with respect to  $g'$  (in fact, with respect to any complete

Kähler metric on  $X'$ ). Finally, we have

$$F \equiv E_1 \cup \dots \cup E_m \subseteq E$$

and, near each point in  $\partial F \cap \partial X'$ , the local defining function  $\varphi - r$  for  $F$  in  $X$  is pluriharmonic with nonvanishing differential. The argument on p. 831 in [NR1] now shows that  $g'$  has bounded geometry of order 2 (in fact, of all orders) along  $F$ .  $\square$

*Remark.* Clearly, either  $X' \setminus K = E_0 \cup E_1 \cup E_2 \cup \dots \cup E_m$  is an ends decomposition or  $E_0 \subseteq X'$  and  $X' \setminus (K \cup E_0) = E_1 \cup E_2 \cup \dots \cup E_m$  is an ends decomposition.

We may now reduce to the bounded geometry case.

**Lemma 4.4.** *To obtain Theorem 3.5, it suffices to prove the theorem for every connected complete Kähler manifold  $X$  which has at least 3 filtered ends and which admits a special ends decomposition into at least  $\min(e(X), 2)$  ends, each of which is hyperbolic of type (BG).*

*Proof.* Given  $(X, g)$  as in the statement of Theorem 3.5, we may choose a special ends decomposition  $X \setminus K = E_1 \cup \dots \cup E_m$  for  $X$  such that  $m \geq \min(e(X), 2)$  and such that, setting  $\Gamma_j = \text{im} [\pi_1(E_j) \rightarrow \pi_1(X)]$  for  $j = 1, \dots, m$ , we have

$$\sum_{j=1}^m [\pi_1(X) : \Gamma_j] \geq 3.$$

According to Lemma 3.7, in order to obtain a proper holomorphic mapping of  $X$  onto a Riemann surface, it suffices to find such a mapping for some nonempty open subset of  $X$ . Applying Lemma 4.2 to each end of type (W) and working on a suitable subdomain in place of  $X$ , we see that we may assume without loss of generality that each of the ends is of type (BG), (RH), or (SP).

Note that the parabolic ends of type (BG) are also of type (SP). Thus, if  $m \geq 2$ , then, for each  $j$ , Theorem 3.6 provides a pluriharmonic function  $\alpha$  on  $X$  such that  $\alpha|_{\overline{E_j}}$  is an exhaustion function if the end is of type (SP) and a bounded exhaustion function if the end is of type (RH). Thus, for  $m \geq 2$ , Lemma 4.3 implies that we may assume that each end is hyperbolic of type (BG) (the condition  $m \geq 2 = \min(e(X), 2)$  will be satisfied automatically for the associated subdomain  $X'$  which replaces  $X$ ).

Thus it suffices to consider the case in which  $e(X) = 1$  and  $E_1$  is a special end of type (RH) or (SP) (i.e.  $X$  is itself a special end of type (RH) or (SP)). We may also choose

the end  $E_1$  to be a  $C^\infty$  domain. For a point  $x_0 \in E_1$ ,  $\Gamma_1 \equiv \text{im} [\pi_1(E_1, x_0) \rightarrow \pi_1(X, x_0)]$  is of index  $\geq 3$ . Thus we may fix a connected covering space  $\Upsilon : \widehat{X} \rightarrow X$  and a point  $y_0 \in \widehat{X}$  such that  $\Upsilon_* \pi_1(\widehat{X}, y_0) = \Gamma_1$ . Hence  $\Upsilon$  maps a neighborhood of the closure of the component  $E$  of  $\widehat{E}_1 = \Upsilon^{-1}(E_1)$  containing  $y_0$  isomorphically onto a neighborhood of  $\overline{E_1}$  and, since  $\#\Upsilon^{-1}(x_0) = [\pi_1(X, x_0) : \Gamma_1] \geq 3$ , we have  $\widehat{E}_1 \setminus E \neq \emptyset$ .

In particular,  $e(\widehat{X}) > 1$ . By Lemma 3.8,  $E$  is a special end of the same type as  $E_1$  (type (RH) or type (SP)) and any component  $F$  of  $\widehat{X} \setminus \overline{E}$  with noncompact closure is either a hyperbolic end (this is the case if, for example, if  $X$  is of type (RH)) or a special end of type (SP) (which is the case if  $X$  is of type (SP) and  $F$  is not hyperbolic). Therefore, by Theorem 3.6, there exists a real-valued pluriharmonic function  $\beta$  on  $\widehat{X}$  whose restriction to  $\overline{E}$  is either an exhaustion function or a bounded exhaustion function. Applying Lemma 4.3 to the function

$$\rho = \beta \circ (\Upsilon|_{\overline{E}})^{-1} - \inf_E \beta$$

on  $\overline{E_1}$ , we get the associated subdomain and Kähler metric  $(X', g')$  in which *any* end is hyperbolic and special of type (BG), (RH), and (W). Choosing an ends decomposition into at least  $\min(e(X'), 2)$  ends, we see that the claim follows.  $\square$

The next two lemmas give the theorem when the manifold has only finitely many filtered ends.

**Lemma 4.5.** *Suppose  $M$  is a connected noncompact manifold and  $M \setminus K = E_1 \cup \dots \cup E_m$  is an ends decomposition such that, setting  $\Gamma_j = \text{im} [\pi_1(E_j) \rightarrow \pi_1(M)]$  for  $j = 1, \dots, m$ , we have*

$$k = \sum_{j=1}^m [\pi_1(M) : \Gamma_j] < \infty.$$

*Then there exists a connected finite covering space  $\widehat{M} \rightarrow M$  with  $e(\widehat{M}) \geq k$ .*

*Proof.* The lifting of  $M \setminus K$  to the universal covering  $\widetilde{M} \rightarrow M$  has  $k$  components, and the action of  $\pi_1(M)$  permutes these components. Thus we get a homomorphism of  $\pi_1(M)$  into the symmetric group on  $k$  objects, and hence the kernel  $\Gamma$  is a normal subgroup of finite index. The quotient  $\widehat{M} = \Gamma \backslash \widetilde{M} \rightarrow M$  is, therefore, a finite covering with at least  $k$  ends.  $\square$

**Lemma 4.6.** *Let  $(X, g)$  be a connected complete Kähler manifold which admits a special ends decomposition  $X \setminus K = E_1 \cup \cdots \cup E_m$  such that, setting  $\Gamma_j = \text{im} [\pi_1(E_j) \rightarrow \pi_1(X)]$  for  $j = 1, \dots, m$ , we have*

$$3 \leq \sum_{j=1}^m [\pi_1(X) : \Gamma_j] < \infty.$$

*Then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

*Proof.* By Lemma 4.5,  $X$  admits a connected finite covering space  $\widehat{X} \rightarrow X$  with at least three ends. Theorem 3.4 and Lemma 3.8 imply that  $\widehat{X}$  admits a proper holomorphic mapping onto a Riemann surface. The claim follows since any normal complex space which is the image of a holomorphically convex complex space under a proper holomorphic mapping is itself holomorphically convex.  $\square$

*Proof of Theorem 3.5.* By Theorem 3.4 and Lemma 4.4, it suffices to consider a connected complete Kähler manifold  $(X, g)$  such that  $\tilde{e}(X) \geq 3$ ,  $m = e(X) \leq 2$ , and we have an ends decomposition  $X \setminus K = E_1 \cup \cdots \cup E_m$  in which each of the ends is hyperbolic and special of type (BG). In particular, since  $m = e(X)$ , every end in  $X$  is hyperbolic and special of type (BG). Thus we may also choose the ends decomposition so that each of the ends  $E_j$  is a smooth domain and, by Lemma 4.6, we may assume that the subgroup  $\Gamma \equiv \text{im} [\pi_1(E_1) \rightarrow \pi_1(X)]$  is of infinite index in  $\pi_1(X)$ . According to Lemma 3.7, it suffices to show that some nonempty open subset of  $X$  admits a proper holomorphic mapping onto a Riemann surface.

Let  $\Upsilon: \widehat{X} \rightarrow X$  be a connected covering space with  $\Upsilon_*\pi_1(\widehat{X}) = \Gamma$  and let  $\hat{g} = \Upsilon^*g$ . Then  $\Upsilon$  is an infinite covering map,  $\Upsilon$  maps a neighborhood of the closure of some component  $E$  of  $\widehat{E}_1 = \Upsilon^{-1}(E_1)$  isomorphically onto a neighborhood of  $\overline{E_1}$ ,  $(\widehat{X}, \hat{g})$  has bounded geometry, and each component of  $X \setminus \partial E$  with noncompact closure is a hyperbolic end (by Lemma 3.8). In particular, it suffices to show that some nonempty open subset of  $\widehat{X}$  admits a proper holomorphic mapping onto a Riemann surface. For then  $\widehat{X}$  and, therefore, some nonempty open subset of  $E \cong E_1 \subset X$ , admits such a mapping.

If the restriction of  $\Upsilon$  to each component of  $\widehat{E}_1$  gives a finite covering space of  $E$ , then  $e(\widehat{X}) = \infty$  and Theorem 3.4 gives the desired proper holomorphic mapping to a Riemann surface. Thus we may assume that there is a component  $Y$  of  $\widehat{E}_1$  such that the restriction  $Y \rightarrow E_1$  is an infinite covering. Since each component of  $X \setminus \partial E$  with

noncompact closure is a hyperbolic end, Theorem 3.6 provides a finite energy pluriharmonic function  $\rho: \widehat{X} \rightarrow (0, 1)$  such that

$$\limsup_{x \rightarrow \infty} \rho|_{\overline{E}}(x) = 1.$$

The  $L^\infty/L^2$ -comparison for local holomorphic functions on a bounded geometry Hermitian manifold implies that  $d\rho$  is bounded. Thus the pluriharmonic function

$$\rho_1 \equiv \rho \circ (\Upsilon|_E)^{-1} \circ \Upsilon|_Y$$

on  $Y$  also has bounded differential. On the other hand,  $d\rho_1$  is *not* in  $L^2$  because the covering space  $Y \rightarrow E_1$  is infinite, so the holomorphic 1-form  $\omega_1 \equiv \partial\rho_1$  and the  $L^2$  holomorphic 1-form  $\omega_2 \equiv \partial\rho|_Y$  are linearly independent. Fixing  $a$  with

$$\max_{\partial E} \rho < a < 1,$$

we get a nonempty connected component  $\Omega$  of  $\{x \in Y \mid a < \rho_1(x)\}$  with  $\overline{\Omega} \subset Y$ . Theorem 2.1 now implies that  $\Omega$  admits a proper holomorphic mapping onto a Riemann surface and the theorem follows.  $\square$

## 5. FURTHER GENERALIZATIONS OF THE CUP PRODUCT LEMMA

In this section, the techniques described in Section 2 will be extended to give several different versions of the cup product lemma. Throughout this section,  $(X, g)$  will denote a connected complete Hermitian manifold of dimension  $n > 1$  with associated real  $(1, 1)$ -form  $\eta$ . We may also form the collection of functions  $\{\tau_r\}_{r>0}$  and the operator  $\mathcal{M}_R$  for  $R > 0$  as in the beginning of Section 2.

Suppose  $(\mathcal{V}, g)$  is a Hermitian inner product space,  $\eta$  is the skew-symmetric real form of type  $(1, 1)$  associated to  $g$ , and  $A$  is a Hermitian symmetric form on  $\mathcal{V}$  with associated skew-symmetric real  $(1, 1)$ -form  $\alpha$ . We will write  $A \geq 0$  and  $\alpha \geq 0$  ( $A > 0$  and  $\alpha > 0$ ) if  $A$  is nonnegative definite (respectively, positive definite). Given a positive integer  $q$ , we will write  $A \geq_{(g,q)} 0$  and  $\alpha \geq_{(g,q)} 0$  ( $A >_{(g,q)} 0$  and  $\alpha >_{(g,q)} 0$ ) if the  $g$ -trace of the restriction of  $A$  to any  $q$ -dimensional vector subspace of  $\mathcal{V}$  is nonnegative (respectively, positive); in other words, for any choice of orthonormal vectors  $e_1, \dots, e_q$  in  $\mathcal{V}$ , we have  $\sum A(e_j, e_j) \geq 0$  (respectively,  $> 0$ ). We will apply the following elementary fact:

**Lemma 5.1.** *Let  $(\mathcal{V}, J)$  be a complex vector space of dimension  $n > 1$ , let  $g$  be a Hermitian inner product on  $\mathcal{V}$  with associated real skew-symmetric  $(1, 1)$ -form  $\eta$ , let  $\alpha$  and  $\beta$  be skew-symmetric real forms of type  $(1, 1)$  on  $\mathcal{V}$ , and let  $\gamma \equiv \alpha \wedge \beta \wedge \eta^{n-2}$ .*

- (a) *If  $\alpha \geq 0$  and  $\beta \geq_{(g, n-1)} 0$ , then  $\gamma$  is nonnegative (i.e.  $\gamma/\eta^n \geq 0$ ).*
- (b) *If  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $\gamma = 0$ , then  $\alpha \wedge \beta = 0$ .*
- (c) *If  $\alpha \geq 0$ ,  $\beta >_{(g, n-1)} 0$ , and  $\gamma = 0$ , then  $\alpha = 0$ .*
- (d) *If  $\alpha = \sqrt{-1}\omega_1 \wedge \overline{\omega_1}$  for some form  $\omega_1$  of type  $(1, 0)$ ,  $\beta \geq 0$ , and  $\gamma = 0$ , then  $\omega_1 \wedge \beta = 0$ .*
- (e) *If  $\alpha = \sqrt{-1}\omega_1 \wedge \overline{\omega_1}$  and  $\beta = \sqrt{-1}\omega_2 \wedge \overline{\omega_2}$  for some pair of forms  $\omega_1$  and  $\omega_2$  of type  $(1, 0)$  and  $\gamma = 0$ , then  $\omega_1 \wedge \omega_2 = 0$ .*

*Remarks.* 1. We used Part (e) (which is easy to verify directly) in the proof of Theorem 0.2.  
 2. Parts (b) and (d) may fail if we only assume that  $\beta \geq_{(g, n-1)} 0$  (in place of  $\beta \geq 0$ ).

*Proof.* Corresponding to any complex basis  $e_1, \dots, e_n$  for  $\mathcal{V}$ , we have

- (i) The real basis  $e_1, f_1 = Je_1, \dots, e_n, f_n = Je_n$ ;
- (ii) The real dual basis  $u_1, v_1 = -u_1 \circ J, \dots, u_n, v_n = -u_n \circ J$ ; and
- (iii) The complex basis  $\zeta_1 = u_1 + \sqrt{-1}v_1, \dots, \zeta_n = u_n + \sqrt{-1}v_n, \overline{\zeta_1}, \dots, \overline{\zeta_n}$  for  $\mathcal{V}_{\mathbb{C}}^*$ .

We may choose the basis so that

$$\eta = \sqrt{-1} \sum \zeta_j \wedge \overline{\zeta_j}, \quad \alpha = \sqrt{-1} \sum A_{ij} \zeta_i \wedge \overline{\zeta_j}, \quad \text{and} \quad \beta = \sqrt{-1} \sum \lambda_j \zeta_j \wedge \overline{\zeta_j};$$

where

$$A_{ij} = \overline{A_{ji}} \quad \forall i, j \quad \text{and} \quad \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

Thus

$$\begin{aligned} \gamma &= (\sqrt{-1})^{n-2} (n-2)! \sum_{1 \leq i < j \leq n} \alpha \wedge \beta \wedge \zeta_1 \wedge \overline{\zeta_1} \wedge \dots \wedge \widehat{\zeta_i \wedge \overline{\zeta_i}} \wedge \dots \wedge \widehat{\zeta_j \wedge \overline{\zeta_j}} \wedge \dots \wedge \zeta_n \wedge \overline{\zeta_n} \\ &= (\sqrt{-1})^n (n-2)! \sum_{1 \leq i < j \leq n} (A_{ii} \lambda_j + A_{jj} \lambda_i) \zeta_1 \wedge \overline{\zeta_1} \wedge \dots \wedge \zeta_n \wedge \overline{\zeta_n} \\ &= 2^n (n-2)! \sum_{i=1}^n \left[ A_{ii} \left( \sum_{j \neq i} \lambda_j \right) \right] u_1 \wedge v_1 \wedge \dots \wedge u_n \wedge v_n. \end{aligned}$$

Part (a) follows immediately.

If  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $\gamma = 0$ , then  $A_{ii} = 0$  whenever  $\lambda_k > 0$  for some  $k \neq i$ . Furthermore, whenever  $A_{ii} = 0$ , we have  $A_{ij} = 0$  for all  $j$  (for example, by the Schwarz inequality).

Hence

$$\alpha \wedge \beta = - \sum_{i,j=1}^n \sum_{k \neq i,j} A_{ij} \lambda_k \zeta_i \wedge \bar{\zeta}_j \wedge \zeta_k \wedge \bar{\zeta}_k = 0$$

as claimed in (b). Similar arguments give (c).

Under the conditions in (d), we have  $A_{ij} = a_i \bar{a}_j$  where  $\omega_1 = \sum a_j \zeta_j$ , and so  $a_i = 0$  whenever  $\lambda_k > 0$  for some  $k \neq i$ . Hence

$$\omega_1 \wedge \beta = \sqrt{-1} \sum_{i=1}^n \sum_{k \neq i} a_i \lambda_k \zeta_i \wedge \zeta_k \wedge \bar{\zeta}_k = 0$$

as claimed.

Under the conditions in (e), we get  $A_{ij} = a_i \bar{a}_j$  for all  $i, j$  and  $0 = \lambda_1 = \dots = \lambda_{n-1} \leq \lambda_n = |b|^2$  where  $\omega_1 = \sum a_j \zeta_j$  and  $\omega_2 = b \zeta_n$ . Hence  $a_1 = \dots = a_{n-1} = 0$  if  $b \neq 0$  and, therefore,

$$\omega_1 \wedge \omega_2 = \sum_{i=1}^{n-1} a_i b \zeta_i \wedge \zeta_n = 0$$

as claimed.  $\square$

It will also be convenient to fix a  $C^\infty$  function  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  such that  $\chi' \geq 0$  and  $\chi'' \geq 0$  on  $\mathbb{R}$ ,  $\chi(t) = 0$  for  $t \leq 0$ , and  $\chi(t) = t - 1$  for  $t \geq 2$ .

**Theorem 5.2.** *Let  $\varphi$  be a real-valued  $C^\infty$  function on  $X$ , let  $\alpha = 2\sqrt{-1}\partial\bar{\partial}\varphi = dd^c\varphi$ , and let  $\beta$  be a  $C^\infty$  real form of type  $(1, 1)$  on  $X$ . Assume that*

- (i) *The real  $(n, n)$ -form  $\gamma \equiv \alpha \wedge \beta \wedge \eta^{n-2}$  satisfies  $\gamma \geq 0$ ;*
- (ii) *We have  $d^c\varphi \wedge d(\beta \wedge \eta^{n-2}) \equiv 0$ ; and*
- (iii) *For some point  $p \in X$ , we have*

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \int_{B(p; 2r) \setminus B(p; r)} |d\varphi|_g |\beta|_g dV_g = 0.$$

*Then the following hold:*

- (a) *We have  $\gamma \equiv 0$  on  $X$ .*
- (b) *At any point  $x \in X$  at which both  $\alpha_x \geq 0$  and  $\beta_x \geq 0$  hold, we have  $(\alpha \wedge \beta)_x = 0$ .*
- (c) *At any point  $x \in X$  at which both  $\alpha_x \geq 0$  and  $\beta_x = \sqrt{-1}\omega \wedge \bar{\omega}$  for some  $\omega \in (T_x^{1,0}X)^*$ , we have  $\alpha_x \wedge \omega = 0$ .*
- (d) *If  $\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2} \geq 0$ , then  $\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2} \equiv 0$ .*



- (e) If  $\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2} \geq 0$ , then, at any point  $x \in X$  at which  $\beta_x \geq 0$  holds, we have  $(\partial\varphi \wedge \beta)_x = 0$ .
- (f) If  $\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2} \geq 0$ , then, at any point  $x \in X$  at which  $\beta_x = \sqrt{-1}\omega \wedge \bar{\omega}$  for some  $\omega \in (T_x^{1,0}X)^*$ , we have  $(\partial\varphi)_x \wedge \omega = 0$ .

*Remarks.* 1. By Lemma 5.1, the condition (i) holds if, for example, at each point  $x \in \text{supp } \alpha \cap \text{supp } \beta$  we have  $\alpha_x \geq 0$  and  $\beta_x \geq_{(g,n-1)} 0$  or we have  $\alpha_x \geq_{(g,n-1)} 0$  and  $\beta_x \geq 0$ .

2. Condition (ii) holds if, for example,  $g$  is Kähler and  $\beta$  is closed off of the set of critical points of  $\varphi$ .

3. The condition  $\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2} \geq 0$  (in Parts (d)–(f)) holds if, for example, at each point  $x \in \text{supp } (d\varphi) \cap \text{supp } \beta$ , we have  $\beta_x \geq_{(g,n-1)} 0$ .

4. The condition (iii) holds if, for example,  $\beta \in L^1$  and, for some constant  $C > 0$ ,  $|d\varphi|_g \leq C(r+1)$  on  $B(p; r)$  for every  $r > 0$ .

5. Clearly, Theorem 0.2 is a special case of the above theorem (one simply takes  $\varphi = \rho$  where  $\rho$  is a pluriharmonic function with  $\text{Re}(\omega_1) = d\rho$ ).

*Proof of Theorem 5.2.* We have

$$\gamma = dd^c\varphi \wedge \beta \wedge \eta^{n-2} = d \left[ d^c\varphi \wedge \beta \wedge \eta^{n-2} \right].$$

Hence, for each  $r > 0$ , Stokes' theorem gives

$$\int_X \tau_r \gamma = - \int_{B(p; 2r) \setminus B(p; r)} d\tau_r \wedge d^c\varphi \wedge \beta \wedge \eta^{n-2}.$$

On the other hand, for some constant  $C = C(n) > 0$ , we have

$$|d\tau_r \wedge d^c\varphi \wedge \beta \wedge \eta^{n-2}|_g \leq \frac{C}{r} |d\varphi|_g |\beta|_g.$$

Thus, for a suitable sequence  $\{r_\nu\}$  with  $r_\nu \rightarrow \infty$ , we get

$$\int_X \gamma \leftarrow \int_X \tau_{r_\nu} \gamma \leq \frac{C}{r_\nu} \int_{B(p; 2r_\nu) \setminus B(p; r_\nu)} |d\varphi|_g |\beta|_g dV_g \rightarrow 0.$$

Since  $\gamma \geq 0$ , we get  $\gamma \equiv 0$  as claimed in (a) and Lemma 5.1 gives (b) and (c) as well.

Assume now that  $\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2} \geq 0$  on  $X$ . Given a constant  $a \in \mathbb{R}$ , the nonnegative  $C^\infty$  function  $\psi = \chi(\varphi - a)$  satisfies

$$dd^c\psi = 2\sqrt{-1}\partial\bar{\partial}\psi = \chi'(\varphi - a)\alpha + 2\sqrt{-1}\chi''(\varphi - a)\partial\varphi \wedge \bar{\partial}\varphi$$

and

$$\begin{aligned}
dd^c\psi^2 &= d(2\psi d^c\psi) = 2\sqrt{-1}\partial\bar{\partial}\psi^2 \\
&= 4\psi\sqrt{-1}\partial\bar{\partial}\psi + 4\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \\
&= 2\chi(\varphi - a)\chi'(\varphi - a)\alpha + \left(\chi(\varphi - a)\chi''(\varphi - a) + [\chi'(\varphi - a)]^2\right) 4\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi
\end{aligned}$$

Thus, by the hypotheses,  $d\theta = \xi \equiv dd^c\psi^2 \wedge \beta \wedge \eta^{n-2} \geq 0$ , where  $\theta \equiv 2\psi d^c\psi \wedge \beta \wedge \eta^{n-2}$ . For every regular value  $R > 1$  of  $\psi$ , let  $\xi_R \geq 0$  be the product of  $\xi$  and the characteristic function of the set  $\{x \in X \mid \psi \leq R\}$ , let  $\psi_R = \mathcal{M}_R(\psi) = \min(\psi, R)$ , and let  $\theta_R$  be the Lipschitz continuous form given by

$$\theta_R \equiv 2\psi_R d^c\psi \wedge \beta \wedge \eta^{n-2}.$$

On the set  $\{x \in X \mid \psi(x) > 1\} \supset \{x \in X \mid \psi(x) \geq R\}$ , we have  $\psi = \varphi - 1$  and hence

$$dd^c\psi \wedge \beta \wedge \eta^{n-2} = \gamma = 0.$$

Therefore  $\xi_R = d\theta_R$  on  $X \setminus \psi^{-1}(R)$ . For each  $r > 0$ , Stokes' theorem gives

$$\int_X \tau_r \xi_R = - \int_{B(p;2r) \setminus B(p;r)} d\tau_r \wedge \theta_R.$$

On the other hand, since  $0 \leq \chi' \leq 1$ , we have  $|d\tau_r \wedge \theta_R|_g \leq 2Cr^{-1}R|d\varphi|_g|\beta|_g$  (where  $C = C(n) > 0$  as before). Letting  $r = r_\nu \rightarrow \infty$  for a suitable sequence  $\{r_\nu\}$ , we get

$$\int_X \xi_R = 0.$$

Since  $\xi_R \geq 0$ , we must have  $\xi_R \equiv 0$  and, letting  $R \rightarrow \infty$ , we get (since  $\gamma \equiv 0$ )

$$\begin{aligned}
0 &= \xi = dd^c\psi^2 \wedge \beta \wedge \eta^{n-2} \\
&= \left(\chi(\varphi - a)\chi''(\varphi - a) + [\chi'(\varphi - a)]^2\right) 4\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2}.
\end{aligned}$$

Now, given a point  $x \in X$ , we may choose  $a \in \mathbb{R}$  so that  $2 < \varphi(x) - a$ . Hence

$$0 = -4^{-1}\sqrt{-1}\xi_x = (\partial\varphi \wedge \bar{\partial}\varphi \wedge \beta \wedge \eta^{n-2})_x,$$

as claimed in (d), and Lemma 5.1 gives (e) and (f). □

We have the following two immediate consequences:

**Corollary 5.3.** *Let  $\varphi$  be a  $C^\infty$  plurisubharmonic function, let  $Z$  be the set of critical points of  $\varphi$ , and let  $\beta$  be a  $C^\infty$  real form of type  $(1, 1)$  on  $X$ . Assume that  $\beta \upharpoonright_{X \setminus Z}$  is closed and nonnegative,  $g \upharpoonright_{X \setminus Z}$  is Kähler, and, for some point  $p \in X$ ,*

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \int_{B(p; 2r) \setminus B(p; r)} |d\varphi|_g |\beta|_g dV_g = 0.$$

*Then*

$$\partial \bar{\partial} \varphi \wedge \beta \equiv 0 \quad \text{and} \quad \partial \varphi \wedge \beta \equiv 0.$$

**Corollary 5.4.** *Let  $\varphi$  be a  $C^\infty$  plurisubharmonic function on  $X$ , let  $Z$  be the set of critical points of  $\varphi$ , and let  $\omega$  be a  $C^\infty$  form of type  $(1, 0)$  on  $X$  such that  $\omega \upharpoonright_{X \setminus Z}$  is closed (hence holomorphic),  $g \upharpoonright_{X \setminus Z}$  is Kähler, and, for some point  $p \in X$ , we have*

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \int_{B(p; 2r) \setminus B(p; r)} |d\varphi|_g |\omega|_g^2 dV_g = 0.$$

*Then*

$$\partial \bar{\partial} \varphi \wedge \omega \equiv 0 \quad \text{and} \quad \partial \varphi \wedge \omega \equiv 0.$$

*Remark.* The above limit inferior is 0 if, for example,  $\omega$  is in  $L^2$  and, for some  $C > 0$ ,  $|d\varphi|_g \leq C(r + 1)$  on  $B(p; 2r)$  for each  $r > 0$ .

**Definition 5.5.** Let  $q$  be a positive integer. A  $C^\infty$  real-valued function  $\varphi$  on an open subset  $\Omega$  of  $X$  is of class  $\mathcal{P}^\infty(g, q)$  (of class  $\mathcal{SP}^\infty(g, q)$ ) if  $\sqrt{-1} \partial \bar{\partial} \varphi \geq_{(g, q)} 0$  (respectively,  $\sqrt{-1} \partial \bar{\partial} \varphi >_{(g, q)} 0$ ).

This class of functions was first introduced by Grauert and Riemenschneider [GR] and has since been applied in several contexts (see, for example, [Siu1], [Wu], [NR3], [Jo], [Fr]). Theorem 5.2 immediately gives the following:

**Corollary 5.6.** *Let  $\varphi \in \mathcal{P}^\infty(g, n - 1)(X)$ , let  $Z$  be the set of critical points of  $\varphi$ , and let  $\omega$  be a  $C^\infty$  form of type  $(1, 0)$  on  $X$  such that  $\omega \upharpoonright_{X \setminus Z}$  is closed (hence holomorphic),  $g \upharpoonright_{X \setminus Z}$  is Kähler, and, for some point  $p \in X$ , we have*

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \int_{B(p; 2r) \setminus B(p; r)} |d\varphi|_g |\omega|_g^2 dV_g = 0.$$

*Then  $\partial \varphi \wedge \omega \equiv 0$  on  $X$ .*

Theorem 2.1 may be considered as a consequence of Corollary 5.4 (or Corollary 5.6) and Theorem 1.4 by fixing a number  $b$  with  $a < b < \sup_{\Omega} \rho_1$  and setting

$$\varphi = \begin{cases} \chi \left( \frac{2\rho_1 - 2a}{b - a} \right) & \text{on } \Omega \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

Similar arguments also give the following theorem which may be viewed both as a variant of Lemma 2.6 of [NR5] (see also Lemma 2.1 of [NR3]) and of Lemma 2.7 of [NR5] and which, in the bounded geometry complete Kähler case, is a generalization of both.

**Theorem 5.7.** *Let  $\rho_1$  and  $\rho_2$  be two real-valued pluriharmonic functions on a nonempty domain  $Y$  in  $X$ . Assume that, for some pair of constants  $a, b$  with  $\inf \rho_1 < a < b < \sup \rho_1$  and some component  $\Omega$  of  $\{x \in Y \mid a < \rho_1(x) < b\}$ , we have the following:*

- (i)  $\overline{\Omega} \subset Y$ ;
- (ii) *The metric  $g \upharpoonright_{\Omega}$  is Kähler;*
- (iii) *The form  $d\rho_2 \upharpoonright_{\Omega}$  is bounded; and*
- (iv)  $\int_{\Omega} |d\rho_j|_g^2 dV_g < \infty$  for  $j = 1, 2$ .

*Then  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$  on  $Y$ . Furthermore, if  $d\rho_1$  and  $d\rho_2$  are linearly independent and  $(X, g)$  has bounded geometry along  $\Omega$ , then there exist a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of  $\Omega$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_{\Omega} = \mathcal{O}_S$  and pluriharmonic functions  $\alpha_1$  and  $\alpha_2$  on  $S$  such that  $\rho_j \upharpoonright_{\Omega} = \Phi^*\alpha_j$  for  $j = 1, 2$ .*

*Proof.* The coarea formula gives

$$\int_a^b \left[ \int_{\rho_1^{-1}(t) \cap \Omega} |d\rho_1|_g d\sigma_g \right] dt = \int_{\Omega} |d\rho_1|_g^2 dV_g < \infty.$$

Thus we may choose regular values  $A$  and  $B$  of  $\rho_1$  in  $\rho_1(\Omega)$  such that  $a < A < B < b$  and such that, for  $M = \rho_1^{-1}(A) \cap \Omega$  and  $N = \rho_1^{-1}(B) \cap \Omega$ , we have

$$\int_M |d\rho_1|_g d\sigma_g < \infty \quad \text{and} \quad \int_N |d\rho_1|_g d\sigma_g < \infty.$$

In particular, we have  $\Theta = \{x \in \Omega \mid A < \rho_1(x) < B\} \neq \emptyset$  and  $\overline{\Theta} \subset \Omega \subset Y$ . The real  $(n, n)$ -form

$$\gamma \equiv (\sqrt{-1}\partial\rho_1 \wedge \bar{\partial}\rho_1) \wedge (\sqrt{-1}\partial\rho_2 \wedge \bar{\partial}\rho_2) \wedge \eta^{n-2}$$

on  $\Omega$  then satisfies

$$0 \leq \gamma = \frac{1}{16} dd^c(\rho_1)^2 \wedge dd^c(\rho_2)^2 \wedge \eta^{n-2} = \frac{1}{4} d(\rho_1 d^c \rho_1) \wedge d(\rho_2 d^c \rho_2) \wedge \eta^{n-2} = d\theta;$$

where  $\theta$  is the  $L^1$  (and  $L^2$ ) form on  $\Omega$  given by

$$\theta = \frac{1}{4} \rho_1 d^c \rho_1 \wedge d(\rho_2 d^c \rho_2) \wedge \eta^{n-2} = \frac{1}{2} \rho_1 d^c \rho_1 \wedge (\sqrt{-1} \partial \rho_2 \wedge \bar{\partial} \rho_2) \wedge \eta^{n-2}.$$

We may form a complete Hermitian metric  $h$  in  $\Omega$  such that  $h \geq g$  on  $\Omega$  and  $h = g$  on a neighborhood of  $\bar{\Theta}$  and, fixing a point  $p \in \Theta$  and applying the Gaffney construction [Ga] and  $C^\infty$  approximation, we get a collection of nonnegative  $C^\infty$  functions  $\{\kappa_r\}_{r>0}$  such that, for each  $r > 0$ , we have  $0 \leq \kappa_r \leq 1$  on  $X$ ,  $\text{supp } \kappa_r \subset B_h(p; 2r) \Subset \Omega$ ,  $\kappa_r \equiv 1$  on  $B_h(p; r)$ , and  $|d\kappa_r|_h \leq 2/r$ .

For each  $r > 0$ , Stokes' theorem gives

$$\begin{aligned} \int_{\Theta} \kappa_r \gamma &= \int_N \kappa_r \theta - \int_M \kappa_r \theta - \int_{[B_h(p; 2r) \setminus B_h(p; r)] \cap \Theta} d\kappa_r \wedge \theta \\ &= \frac{B}{4} \int_N \kappa_r d^c \rho_1 \wedge d(\rho_2 d^c \rho_2) \wedge \eta^{n-2} - \frac{A}{4} \int_M \kappa_r d^c \rho_1 \wedge d(\rho_2 d^c \rho_2) \wedge \eta^{n-2} \\ &\quad - \int_{[B_h(p; 2r) \setminus B_h(p; r)] \cap \Theta} d\kappa_r \wedge \theta \\ &= \frac{B}{4} \int_N d\kappa_r \wedge d^c \rho_1 \wedge \rho_2 d^c \rho_2 \wedge \eta^{n-2} - \frac{A}{4} \int_M d\kappa_r \wedge d^c \rho_1 \wedge \rho_2 d^c \rho_2 \wedge \eta^{n-2} \\ &\quad - \int_{[B_h(p; 2r) \setminus B_h(p; r)] \cap \Theta} d\kappa_r \wedge \theta \end{aligned}$$

Since  $|d\rho_2|_h \leq |d\rho_2|_g$ ,  $|d\rho_2|_h$  is bounded on  $\Omega$  and hence  $\rho_2$  must have at most linear growth; i.e. for some constant  $C > 0$ , we have  $|\rho_2| \leq C(r+1)$  on  $B_h(p; r)$  for every  $r > 0$ . Moreover,  $|d^c \rho_1|_g$  is in  $L^1$  on  $M$  and  $N$  by the choice of  $A$  and  $B$ . Thus we may apply the dominated convergence theorem as  $r \rightarrow \infty$  to get

$$\int_{\Theta} \gamma = 0.$$

Since  $\gamma \geq 0$ , we get  $\gamma \equiv 0$  on  $\Theta$  and Lemma 5.1 implies that  $\partial \rho_1 \wedge \partial \rho_2 \equiv 0$  on  $\Theta$  and, therefore, on  $Y$ .

If  $d\rho_1$  and  $d\rho_2$  are linearly independent and  $(X, g)$  has bounded geometry along  $\Omega$ , then we may apply Corollary 1.5 to get the desired proper holomorphic mapping to a Riemann surface  $\Phi: \Omega \rightarrow S$ .  $\square$

In particular, we get the following weak version of Lemma 2.1 of [NR3] (see also Lemma 2.6 of [NR5]) which suffices for the proof of the main result of [NR5]:

**Corollary 5.8.** *Let  $\rho_1$  and  $\rho_2$  be two real-valued pluriharmonic functions on  $X$ . Assume that  $\rho_1$  has a nonempty compact fiber  $F$  for which  $g$  is Kähler on a neighborhood of  $F$ . Then  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$  on  $X$ . Furthermore, if  $d\rho_1$  and  $d\rho_2$  are linearly independent, then there exist a proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of a neighborhood  $\Omega$  of  $F$  in  $X$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$  and pluriharmonic functions  $\alpha_1$  and  $\alpha_2$  on  $S$  such that  $\rho_j|_\Omega = \Phi^*\alpha_j$  for  $j = 1, 2$ .*

## 6. $L^2$ CASTELNUOVO-DE FRANCHIS FOR AN END

In this section, the arguments of Section 1 are extended in order to obtain the following version of the  $L^2$  Castelnuovo-de Franchis theorem which is applied in [NR6] and is also of separate interest:

**Theorem 6.1.** *Let  $(X, g)$  be a connected noncompact complete Hermitian manifold, let  $E$  be a special end of type (BG) in  $X$ , and let  $\omega_1$  and  $\omega_2$  be linearly independent closed holomorphic 1-forms on  $E$  such that  $g|_E$  is Kähler,  $\omega_1$  is in  $L^2$ , and  $\omega_1 \wedge \omega_2 \equiv 0$  on  $E$ . Then there exist a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of a nonempty open subset  $\Omega$  of  $E$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$  and holomorphic 1-forms  $\theta_1$  and  $\theta_2$  on  $S$  such that  $\omega_j|_\Omega = \Phi^*\theta_j$  for  $j = 1, 2$ .*

*Remark.* In particular, by Lemma 3.7, if, in addition,  $g$  is Kähler and  $X$  admits a special ends decomposition, then  $X$  admits a proper holomorphic mapping onto a Riemann surface.

**Corollary 6.2.** *Let  $(X, g)$  be a connected noncompact complete Hermitian manifold, let  $E$  be a special end of type (BG) in  $X$ , and let  $\rho_1$  and  $\rho_2$  be two real-valued pluriharmonic functions on  $E$  such that  $g|_E$  is Kähler,  $d\rho_1$  and  $d\rho_2$  are linearly independent,  $\rho_1$  has finite energy, and  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$  on  $E$ . Then there exist a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of a nonempty open subset  $\Omega$  of  $E$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$  and real-valued pluriharmonic functions  $\alpha_1$  and  $\alpha_2$  on  $S$  such that  $\rho_j|_\Omega = \Phi^*\alpha_j$  for  $j = 1, 2$ .*

*In particular, if there exists a nonconstant holomorphic function with finite energy on  $E$ , then there exists a surjective proper holomorphic mapping  $\Phi: \Omega \rightarrow S$  of a nonempty open subset  $\Omega$  of  $E$  onto a Riemann surface  $S$  with  $\Phi_*\mathcal{O}_\Omega = \mathcal{O}_S$ .*

For the proof of Theorem 6.1, we will show that, under the assumption that the associated holomorphic mapping to  $\mathbb{P}^1$  has no nonempty compact levels, one gets a holomorphic function which is defined in  $E$  outside a relatively compact neighborhood of  $\partial E$  and which vanishes at infinity. In particular, the end is then parabolic by the following observation of J. Wang (cf. Lemma 1.3 of [NR3]):

**Lemma 6.3.** *Let  $(X, g)$  be a complete Hermitian manifold and let  $E$  be an end in  $X$  such that  $g|_E$  is Kähler. Assume that there exists a nonconstant holomorphic function  $h$  on a neighborhood of  $\overline{E}$  such that  $\lim_{x \rightarrow \infty} h|_{\overline{E}}(x) = 0$ . Then  $E$  is a parabolic end.*

*Proof.* The function  $\varphi = -\log|h|^2: \overline{E} \rightarrow (-\infty, \infty]$  is superharmonic on  $E$  and  $\varphi(x) \rightarrow \infty$  as  $x \rightarrow \infty$  in  $\overline{E}$ . In particular, we may assume that  $\varphi$  is positive. Suppose  $\alpha$  is a nonnegative bounded subharmonic function on  $X$  which vanishes on  $X \setminus E$ . Given  $\epsilon > 0$ , we have  $\epsilon\varphi > 0 = \alpha$  on  $\partial E$  and  $\epsilon\varphi > \sup \alpha$  on the complement in  $E$  of a sufficiently large compact subset of  $X$ . It follows that  $0 \leq \alpha < \epsilon\varphi$  on  $\overline{E}$  for every  $\epsilon > 0$  and, therefore, that  $\alpha \equiv 0$ . Thus  $E$  does not admit an admissible subharmonic function and hence  $E$  is a parabolic end.  $\square$

We will apply the following lemma which is a consequence of the work of Grauert and Riemenschneider [GR] (for a relatively compact domain  $E$ ), of Gromov [Gro2] and of Li [L] (for  $E = X$ ), and of Siu [Siu1] (for a harmonic mapping of a relatively compact domain into a manifold satisfying certain curvature conditions).

**Lemma 6.4** (Grauert-Riemenschneider, Li, Siu (see Lemma 3.2 of [NR3])). *Let  $(X, g)$  be a connected complete Hermitian manifold of dimension  $n$ , let  $E$  be a (not necessarily relatively compact) domain with smooth compact (possibly empty) boundary in  $X$ , let  $\varphi$  be  $C^\infty$  real-valued function on  $X$  such that  $d\varphi \neq 0$  at every point in  $\partial E$  and such that  $E = \{x \in X \mid \varphi(x) < 0\}$ , and, for each point  $x \in \partial E$ , let*

$$\tau(x) = \operatorname{tr} \left[ \mathcal{L}(\varphi)|_{T_x^{1,0}(\partial E)} \right].$$

*Assume that  $g|_E$  is Kähler and that  $\tau \geq 0$  on  $\partial E$ . Then we have the following:*

- (a) *If  $\beta$  is a  $C^\infty$  function on  $\overline{E}$  such that  $\beta$  is harmonic on  $E$ ,  $\beta$  satisfies the tangential Cauchy-Riemann equation  $\bar{\partial}_b \beta = 0$  on  $\partial E$ , and there is a sequence of positive real*

numbers  $R_m \rightarrow \infty$  and a point  $p \in X$  such that

$$\lim_{m \rightarrow \infty} \frac{1}{R_m^2} \|\nabla \beta\|_{L^2(B_p(R_m) \cap E)}^2 = \lim_{m \rightarrow \infty} \frac{1}{R_m^2} \int_{B_p(R_m) \cap E} |\nabla \beta|^2 dV = 0,$$

then  $\beta$  is pluriharmonic on  $E$ .

(b) If  $E$  is a hyperbolic end of  $X$ , then  $\tau \equiv 0$  on  $\partial E$ .

As suggested by the above lemma, functions of class  $\mathcal{SP}^\infty(g, q)$  will play a role in the proof of the parabolic case of the theorem. The following fact is contained implicitly in the work of Richberg [Ri], Greene and Wu [GreW], Ohsawa [O], Coltoiu [Col], and Demailly [Dem2] (see [NR2]):

**Proposition 6.5** (Richberg, Greene-Wu, Ohsawa, Coltoiu, Demailly). *Suppose  $(X, g)$  is a Hermitian manifold of dimension  $n > 1$  and  $Y$  is a nowhere dense analytic subset with no compact irreducible components. Then there exists a  $C^\infty$  exhaustion function  $\varphi$  on  $X$  which is of class  $\mathcal{SP}^\infty(g, n-1)$  on a neighborhood of  $Y$  in  $X$ .*

The following proposition will give the parabolic case of the theorem and is also of separate interest:

**Proposition 6.6.** *Let  $(X, g)$  be a connected noncompact complete Hermitian manifold and let  $E$  be a special end of type (BG) in  $X$ . Assume that  $g|_E$  is Kähler and that there exists a nonconstant holomorphic function  $h$  on  $E$  which vanishes at infinity in  $X$ . Then there exists a proper holomorphic mapping of some nonempty open subset of  $E$  onto a Riemann surface.*

*Remarks.* 1. The end  $E$  is necessarily parabolic by Lemma 6.3.

2. The above proposition also holds if  $E$  is special of type (W) instead of type (BG); as the proof below together with Proposition 2.3 of [NR2] shows.

*Proof of Proposition 6.6.* We may assume without loss of generality that  $n = \dim X > 1$ . The idea of the proof is to construct an end with a defining function of class  $\mathcal{P}^\infty(g, n-1)$  at the compact boundary and to apply Lemma 6.4. Let  $Z = h^{-1}(0) \subset E$  and fix open subsets  $\Omega_1, \Omega_2$ , and  $\Omega_3$  such that

$$\partial E \subset \Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset X$$



and  $Z \cap \Omega_3 \setminus \overline{\Omega}_1$  is empty or has no compact irreducible components (one can form these sets by choosing open sets  $\Omega_1 \Subset \Omega'_3 \Subset X$  and setting  $\Omega_3 = \Omega'_3 \setminus F$ , where  $F$  is a finite subset of  $E \setminus \overline{\Omega}_1$  which contains a point in  $Z_0 \cap \Omega'_3 \setminus \overline{\Omega}_1$  for each irreducible component  $Z_0$  of  $Z$  which meets  $\Omega'_3 \setminus \overline{\Omega}_1$ ). By Proposition 6.5 (Richberg, Greene-Wu, Ohsawa, Coltoiu, Demailly), there exists a neighborhood  $V$  of  $Z \cap \Omega_3 \setminus \overline{\Omega}_1$  in  $E \cap \Omega_3 \setminus \overline{\Omega}_1$  and a positive function  $\psi \in \mathcal{SP}^\infty(g, n-1)(V)$  which exhausts  $Z \cap \Omega_3 \setminus \overline{\Omega}_1$ . We may also fix constants  $a$  and  $b$  with

$$a > b > \max_{Z \cap \partial\Omega_2} \psi \quad (a > b > 0 \text{ if } Z \cap \partial\Omega_2 = \emptyset).$$

After shrinking  $V$ , we may assume that there is a neighborhood of  $\partial(\Omega_3 \setminus \overline{\Omega}_1)$  in  $X$  such that  $\psi > a$  at points in  $V$  which lie in this neighborhood. Therefore, for  $\epsilon > 0$  sufficiently small, the set

$$\{x \in V \mid |h(x)| \leq \epsilon \text{ and } \psi(x) \leq a\}$$

is compact (and possibly empty) and the compact (and possibly empty) set

$$\{x \in \partial\Omega_2 \mid |h(x)| \leq \epsilon\}$$

is contained in  $\{x \in V \mid \psi(x) < b\}$ . Choosing a  $C^\infty$  nondecreasing convex function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  which vanishes on the interval  $(-\infty, -\log(a-b)]$  and which approaches  $+\infty$  at  $+\infty$ , we obtain a function  $\varphi$  of class  $\mathcal{P}^\infty(g, n-1)$  on the open set

$$\Omega = \{x \in E \setminus \Omega_2 \mid |h(x)| < \epsilon\} \cup \{x \in V \cap \Omega_2 \mid |h(x)| < \epsilon \text{ and } \psi(x) < a\}$$

by defining

$$\varphi(x) = \begin{cases} -\log(\epsilon^2 - |h(x)|^2) & \text{if } x \in E \cap \Omega \setminus \Omega_2 \\ \chi(-\log(a - \psi(x)) - \log(\epsilon^2 - |h(x)|^2)) & \text{if } x \in E \cap \Omega \cap \Omega_2 \end{cases}$$

Moreover,  $\varphi \geq -\log \epsilon^2$  on  $\Omega$ , and, since  $h$  vanishes at infinity in  $X$ ,  $\Omega$  is not relatively compact in  $X$ , the boundary  $\partial\Omega$  in  $X$  (and in  $E$ ) is a nonempty compact subset of  $E$ ,  $\varphi \rightarrow \infty$  at  $\partial\Omega$ , and  $\varphi \rightarrow -\log \epsilon^2$  at infinity in  $X$ . Therefore, if  $c$  is a regular value of  $\varphi$  with  $c > -\log \epsilon^2$  and  $E_0$  is a connected component of the set  $\{x \in \Omega \mid \varphi(x) < c\}$  with noncompact closure in  $X$ , then  $E_0$  is a special end of type (BG) in  $X$  with smooth nonempty (compact) boundary and  $E_0$  admits a defining function of class  $\mathcal{P}^\infty(g, n-1)$  (on a neighborhood of  $\overline{E}_0$ ) with nonvanishing differential at each boundary point.

As in the proof of Theorem 2.6 of [NR1], one may apply a theorem of Nakai [Na1], [Na2] and a theorem of Sullivan [Sul] to obtain a continuous function  $\rho$  on  $\overline{E}_0$  such that  $\rho$  is

harmonic on  $E_0$ ,  $\rho = 0$  on  $\partial E_0$ ,  $\rho(x) \rightarrow \infty$  at infinity, and the  $L^2$  norm of  $\nabla \rho$  on a ball of radius  $R$  is equal to  $o(R)$ . Therefore  $\rho$  is pluriharmonic by Lemma 6.4 and  $\rho$  has a (nonempty) compact fiber in  $E_0$ . The pair of pluriharmonic functions  $\rho$  and  $\operatorname{Re} h|_{E_0}$  must have linearly independent differentials on  $E_0$  (since  $\rho \rightarrow \infty$  while  $\operatorname{Re} h \rightarrow 0$  at infinity), so Corollary 5.8 now gives the claim.  $\square$

**Lemma 6.7.** *Let  $(X, g)$  be a connected noncompact complete Hermitian manifold, let  $E$  be an end of  $X$ , let  $h: E \rightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$  be a holomorphic mapping with no nonempty compact levels, let  $C = \{x \in E \mid (h_*)_x = 0\}$  be the set of critical points, let  $Z$  be the union of all connected components  $A$  of  $C$  for which  $\bar{A}$  is noncompact and meets  $\partial E$ , and let  $r \in (0, \infty)$ . Assume that  $g|_E$  is Kähler, the extended real line  $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\} \subset \mathbb{P}^1$  is contained in  $\mathbb{P}^1 \setminus h(Z)$ ,  $(\mathbb{R} \setminus \{0\}) \cup \{\infty\} \subset \mathbb{P}^1 \setminus h(C)$  (the set of regular values), and*

$$h^{-1}(t) \Subset X \quad \forall t \in (0, r).$$

*Then, for some relatively compact open subset  $\Omega$  of  $X$  containing  $\partial E$ , the function*

$$t \mapsto \operatorname{vol} (h^{-1}(t) \setminus \bar{\Omega})$$

*is bounded on the interval  $(0, r)$ .*

*Proof.* Let  $n = \dim X$  and let  $\eta$  be the real  $(1, 1)$ -form associated to  $g$ . The idea is to apply the Stokes theorem arguments of Stoll [St] while keeping track of the boundary integrals. The set  $M = h^{-1}(\mathbb{R} \cup \{\infty\})$  is a (properly embedded) real analytic subset of  $E$  and the subset  $M \setminus C$  is a (properly embedded) oriented (with nonvanishing  $(2n - 1)$ -form  $(h^* d\theta) \wedge \eta^{n-1}$ ) real analytic submanifold of dimension  $2n - 1$  in  $E \setminus C$  (assuming, as we may, that  $M$  is nonempty). We also have  $M \cap C = M \cap h^{-1}(0) \cap C \setminus Z$ .

We may choose a real-valued  $C^\infty$  function  $\rho$  on  $X$ , a relatively compact neighborhood  $W$  of  $\partial E$  in  $X$ , and a constant  $\epsilon > 0$  such that  $[-\epsilon, \epsilon] \subset \rho(E)$ , each  $\xi \in [-\epsilon, \epsilon]$  is a regular value for  $\rho$  and for  $\rho|_{M \setminus C}$ ,  $\rho < -\epsilon$  on  $\partial E$ ,  $\rho > \epsilon$  on  $X \setminus W$ , and  $|\rho| > \epsilon$  on  $C \setminus Z$ . To see this, we fix a relatively compact neighborhood  $V$  of  $\partial E$  in  $X$ , we let  $A$  be the union of all the relatively compact (in  $X$ ) connected components of  $C$  whose closures either meet  $\partial E$  or are contained in  $V$ , and we let  $B = C \setminus (A \cup Z)$ . The closed set  $A \cup \partial E$  is compact. For only finitely many connected components of  $C$  can meet  $\partial V$  and, of the other connected components, those contained in  $A$  must then be contained in  $V$ . The set  $B$  must be closed in  $X$ , since each of the finitely many connected components of  $B$  which meet  $\partial V$  is closed

in  $X$  ( $A \cup Z$  contains all of the non-closed connected components of  $C$ ) and the remaining connected components are contained in  $X \setminus \overline{V}$  ( $A$  contains all of the connected components contained in  $V$ ). Thus we may choose a relatively compact neighborhood  $W$  of  $A \cup \partial E$  in  $X \setminus B$  and a nonnegative  $C^\infty$  function  $\alpha$  on  $X$  such that  $\alpha \equiv 0$  on  $A \cup \partial E$ ,  $\alpha > 1$  on  $X \setminus W$ , and  $[0, 1] \subset \alpha(E)$ . In particular, we have  $\alpha^{-1}((0, 1)) \cap C \cap M \subset W \cap C \cap M \subset Z \cap M = \emptyset$ . Choosing a number  $s \in (0, 1)$  which is a regular value of both  $\alpha$  and  $\alpha|_{(M \setminus C)}$ , we see that the neighborhood  $W$ , the function  $\rho = \alpha - s$ , and any sufficiently small  $\epsilon > 0$  have the required properties.

Let  $H$  be the compact subset of the manifold  $M \setminus C$  given by

$$H = h^{-1}([0, r]) \cap \rho^{-1}([-\epsilon, \epsilon]),$$

and, for each  $\xi \in [-\epsilon, \epsilon]$ , let  $H_\xi$  be the fiber  $H \cap \rho^{-1}(\xi)$ ; a compact subset of the submanifold  $\rho^{-1}(\xi) \cap M = \rho^{-1}(\xi) \cap (M \setminus C)$  of  $M \setminus C$ . There is then a constant  $u > 0$  giving the uniform bound

$$\left| \int_{H_\xi} \eta^{n-1} \upharpoonright_{\rho^{-1}(\xi) \cap (M \setminus C)} \right| < u \quad \forall \xi \in [-\epsilon, \epsilon]$$

(to see this, one first obtains the local inequality by considering local coordinates of the form  $(\rho, x_2, \dots, x_{2n-1})$  in  $M \setminus C$  and then covers  $H$  by finitely many such coordinate neighborhoods). For each  $\xi \in [-\epsilon, \epsilon]$ , let  $\Omega_\xi = \{x \in X \mid \rho(x) < \xi\}$ ; a relatively compact neighborhood of  $\partial E$  in  $X$ .

Given  $a$  and  $b$  with  $0 < a < b < r$ , we may choose a regular value  $\xi = \xi(a, b) \in (-\epsilon, \epsilon)$  for  $\rho|_{h^{-1}(a) \cup h^{-1}(b)}$ . The set  $D = h^{-1}((a, b)) \setminus \overline{\Omega}_\xi$  is then a relatively compact open subset of  $M \setminus C$  with boundary

$$\partial D = \overline{\Gamma}_0 \cup \overline{\Lambda} \cup \overline{\Gamma}_1;$$

where  $\Gamma_0 = h^{-1}(a) \setminus \overline{\Omega}_\xi$ ,  $\Gamma_1 = h^{-1}(b) \setminus \overline{\Omega}_\xi$ , and  $\Lambda = h^{-1}((a, b)) \cap \partial \Omega_\xi$ . For if  $t \in (0, r)$ , then  $(\partial E) \cup h^{-1}(t) \subset U \Subset X$  for some open set  $U$  and  $h^{-1}(I) \cap \partial U = \emptyset$  for some open interval  $I$  with  $t \in I \subset (0, r)$ . On the other hand, each nonempty level of  $h$  over a point in  $I$  is both relatively compact in  $X$  and noncompact, and, therefore, must meet  $U$ . Hence  $h^{-1}(I) \subset U$  and it follows that  $D \Subset M$ . We have  $\overline{D} \subset h^{-1}([a, b]) \subset M \setminus C$ , so  $D \Subset M \setminus C$ . Moreover  $D$  is smooth at each boundary point not in the ‘‘corners’’  $\overline{\Gamma}_0 \cap \overline{\Lambda}$  and  $\overline{\Lambda} \cap \overline{\Gamma}_1$ . Applying Stokes’ theorem to the closed form  $\eta^{n-1}$ , we get

$$\int_{\Gamma_1} \eta^{n-1} - \int_{\Gamma_0} \eta^{n-1} = \int_{\Lambda} \eta^{n-1}$$

(where we take the orientations associated to the complex structure on  $\Gamma_0$  and  $\Gamma_1$  and the orientation outward from  $\Omega_\xi$  on  $\Lambda$ , and we have used the choice of  $\xi$  as a regular value for  $\rho \upharpoonright_{h^{-1}(a) \cup h^{-1}(b)}$ ). On the other hand, we have  $\Lambda \subset H_\xi$ , so the absolute value of the integral on the right-hand side of the above equality is bounded above by the constant  $u$  which does not depend on the choice of  $a$  and  $b$ . Thus

$$0 \leq \int_{h^{-1}(a) \setminus \overline{\Omega}_\epsilon} \eta^{n-1} \leq \int_{\Gamma_0} \eta^{n-1} \leq u + \int_{\Gamma_1} \eta^{n-1} \leq u + \int_{h^{-1}(b) \setminus \overline{\Omega}_{-\epsilon}} \eta^{n-1}$$

and, similarly,

$$0 \leq \int_{h^{-1}(b) \setminus \overline{\Omega}_\epsilon} \eta^{n-1} \leq u + \int_{h^{-1}(a) \setminus \overline{\Omega}_{-\epsilon}} \eta^{n-1}.$$

Fixing  $c \in (0, r)$  and setting  $\Omega = \Omega_\epsilon$ , we see that

$$\int_{h^{-1}(t) \setminus \overline{\Omega}} \eta^{n-1} \leq R \quad \forall t \in (0, r),$$

where

$$R = u + \int_{h^{-1}(c) \setminus \overline{\Omega}_{-\epsilon}} \eta^{n-1}.$$

The lemma now follows.  $\square$

**Lemma 6.8.** *Let  $(X, g)$  be a connected noncompact complete Hermitian manifold, let  $E$  be a special end of type  $(BG)$  in  $X$ , and let  $A$  be an analytic subset of  $E$  with positive dimension at each point. Then  $A \Subset X$  if and only if  $A \setminus \overline{\Omega}$  has finite volume for some (and, therefore, for every) relatively compact neighborhood  $\Omega$  of  $\partial E$  in  $X$ .*

*Proof.* Suppose  $A$  is an analytic subset of  $E$  with  $\dim_x A > 0$  for each  $x \in A$ . We proceed as in the first part of the proof of Theorem 0.1. Given a relatively compact neighborhood  $\Omega$  of  $\partial E$  in  $X$ , Lelong's monotonicity formula (see 15.3 in [Chi]) implies that there is a constant  $c > 0$  such that each point  $p \in E \setminus \Omega$  has a neighborhood  $U_p \Subset E$  for which  $\text{diam}(U_p) < 1$  and  $\text{vol}(D \cap U_p) \geq c$  for every connected analytic set  $D$  of positive dimension in  $E$  with  $p \in D$ . Therefore, if  $A \setminus \overline{\Omega}$  has finite volume, then  $\overline{A}$  must be compact. The claim now follows easily.  $\square$

**Lemma 6.9.** *Let  $(X, g)$  be a connected noncompact complete Hermitian manifold, let  $E$  be a special end of type  $(BG)$  in  $X$ , and let  $h: E \rightarrow \mathbb{P}^1$  be a nonconstant holomorphic mapping such that the levels over almost every point in  $\mathbb{P}^1$  have finite volume and every nonempty level is noncompact.*

- (a) *If  $F$  is a fiber of  $h$  with noncompact closure in  $X$ , then  $F$  has a connected component  $L$  such that  $\overline{L}$  is noncompact and  $\overline{L} \cap \partial E \neq \emptyset$ .*
- (b) *The set  $Q = \{ \zeta \in \mathbb{P}^1 \mid h^{-1}(\zeta) \Subset X \} = \{ \zeta \in \mathbb{P}^1 \mid L \Subset X \text{ for each level } L \text{ over } \zeta \}$  (the second equality follows from (a)) is open and the inverse image of any compact subset of  $Q$  is relatively compact in  $X$ .*

*Proof.* For the proof of (a), suppose  $\zeta_0 \in \mathbb{P}^1$  is a point for which the corresponding fiber  $F = h^{-1}(\zeta_0)$  is not relatively compact in  $X$ . Fix a relatively compact neighborhood  $\Omega$  of  $\partial E$  in  $X$  and let  $L_1, \dots, L_m$  be the (finitely many) connected components of  $F$  which meet  $\partial\Omega$ . Since  $h$  has no nonempty compact levels, the closure of any connected component of  $F$  which does not meet  $\partial\Omega$  must either lie in  $\Omega$  and meet  $\partial E$  or lie in  $E \setminus \overline{\Omega}$ . Thus the union  $H$  of all of the connected components of  $F$  which are closed in  $X$  is itself a closed set in  $X$  which is contained in  $L_1 \cup \dots \cup L_m \cup (E \setminus \overline{\Omega})$ . Therefore, by replacing  $\Omega$  with a relatively compact neighborhood of  $\partial E$  in  $X \setminus H$ , we may assume that  $\overline{L}_i \cap \partial E \neq \emptyset$  for  $i = 1, \dots, m$ .

Now if the levels  $L_1, \dots, L_m$  are relatively compact in  $X$ , then, by replacing  $\Omega$  with a relatively compact neighborhood of the compact set  $\overline{\Omega} \cup L_1 \cup \dots \cup L_m$  in  $X \setminus H$ , we may assume that  $F \cap \partial\Omega = \emptyset$ ; that is,  $\zeta_0$  lies in the complement  $V \equiv \mathbb{P}^1 \setminus h(E \cap \partial\Omega)$  of the compact set  $h(E \cap \partial\Omega)$ . On the other hand,  $F$  meets  $E \setminus \overline{\Omega}$  and  $h$  is an open mapping, so  $h(E \setminus \overline{\Omega})$  is a neighborhood of  $\zeta_0$  in  $\mathbb{P}^1$ . Since the levels of  $h$  over almost every point in  $\mathbb{P}^1$  have finite volume, Lemma 6.8 implies that there exists a point  $\zeta \in V$  such that  $h^{-1}(\zeta)$  has a relatively compact connected component  $L$  which meets  $E \setminus \overline{\Omega}$ . Since  $h$  has no nonempty compact levels,  $L$  must meet  $\partial\Omega$  and we have arrived at a contradiction. Thus there is a connected component of  $F$  which is not relatively compact and which is not closed in  $X$ , and (a) is proved.

For the proof of (b), given a point  $\zeta_0 \in Q = \{ \zeta \in \mathbb{P}^1 \mid h^{-1}(\zeta) \Subset X \}$ , we may choose a relatively compact neighborhood  $\Omega$  of the compact set  $h^{-1}(\zeta_0) \cup \partial E$  in  $X$ . The open set  $V = \mathbb{P}^1 \setminus h(E \cap \partial\Omega)$  is then a neighborhood of  $\zeta_0$ . If  $\zeta \in \mathbb{P}^1$  and  $F = h^{-1}(\zeta)$  meets  $E \setminus \Omega$ , then (a) implies that  $F$  has a (noncompact) connected component  $L$  such that either  $\overline{L}$  is noncompact and meets  $\partial E$  or  $L$  is relatively compact in  $X$  and meets  $E \setminus \Omega$ . In either case,  $L$  meets both  $\Omega$  and  $E \setminus \Omega$ , and hence  $L$  meets  $\partial\Omega$ . It follows that  $h^{-1}(V) \subset \Omega \Subset X$ . In particular,  $V \subset Q$  and hence  $Q$  is open. Moreover if  $K \subset Q$  is a compact subset, then

one may cover  $K$  by finitely many such neighborhoods  $V$  with relatively compact inverse image and, therefore,  $h^{-1}(K) \Subset X$ . Thus (b) is proved.  $\square$

**Lemma 6.10.** *Suppose  $(X, g)$  is a connected noncompact complete Hermitian manifold,  $E$  is a special end of type (BG) in  $X$ ,  $h: E \rightarrow \mathbb{P}^1$  is a nonconstant holomorphic mapping,  $C$  is the set of critical points of  $h$ , and  $Z$  is the union of all connected components  $A$  of  $C$  for which  $\bar{A}$  is noncompact and  $\bar{A} \cap \partial E \neq \emptyset$ . Assume that  $g|_E$  is Kähler, the levels of  $h$  over almost every point in  $\mathbb{P}^1$  have finite volume, and every nonempty level is noncompact. Then the fiber of  $h$  over every point in  $\mathbb{P}^1 \setminus h(Z)$  is relatively compact in  $X$ .*

*Proof.* We must show that each point  $\zeta_0 \in \mathbb{P}^1 \setminus h(Z)$  lies in the open set

$$Q \equiv \{ \zeta \in \mathbb{P}^1 \mid h^{-1}(\zeta) \Subset X \}.$$

As in the proof of Theorem 0.1, the idea is to consider converging sequences of fibers with uniformly bounded volume (as in [Gro2], [ArBR], and [ABCKT]). We first observe that, by Lemma 6.8 and Lemma 6.9,  $\mathbb{P}^1 \setminus Q$  is a closed set of measure 0. Therefore, since the set of critical values of  $h$  is countable, we may choose an extended real line  $\ell$  through  $\zeta_0$  such that  $\ell \setminus \{\zeta_0\} \subset \mathbb{P}^1 \setminus h(C)$  and  $\ell \setminus Q$  is a (closed) set of (real 1-dimensional) measure 0. If  $\ell$  is not contained in  $Q$ , then there is an open segment  $I$  in  $\ell \cap Q$  which has a boundary point  $a \in \ell$  and another boundary point not equal to  $a$ . Fixing a sufficiently large relatively compact neighborhood  $\Omega$  of  $\partial E$  in  $X$  and applying Lemma 6.7, we get a constant  $v$  such that

$$\text{vol}(h^{-1}(t) \setminus \bar{\Omega}) < v \quad \forall t \in I.$$

As in the proof of Lemma 6.8, this volume bound also gives a bound on the diameter relative to the distance function on  $X$ . For Lelong's monotonicity formula (see 15.3 in [Chi]) implies that there is a constant  $c > 0$  such that each point  $p \in E \setminus \Omega$  has a neighborhood  $U_p \Subset E$  for which  $\text{diam}(U_p) < 1/2$  and  $\text{vol}(D \cap U_p) \geq c$  for every connected analytic set  $D$  of positive dimension in  $E$  with  $p \in D$ . We may also fix a point  $O \in X$  and a constant  $R > 0$  such that  $\Omega \subset B(O; R)$ . If some level  $L$  of  $h$  over a point  $t \in I \subset Q$  meets  $E \setminus B(O; R + k)$  for some positive integer  $k$ , then, since  $L$  also meets  $\Omega$ ,  $L$  meets  $\partial B(O; R + j)$  for all  $j = 1, \dots, k$ . The volume estimate then implies that

$$v > \text{vol}(L \setminus \bar{\Omega}) \geq kc.$$

Thus  $h^{-1}(t) \subset B = B(O; R + (v/c) + 1) \Subset X$  for all  $t \in I$ . But  $h^{-1}(a)$  is not relatively compact in  $X$  and  $h$  is an open mapping, so  $h(E \setminus \overline{B})$  is an open set which contains  $a$  and, therefore, a point  $t \in I$ . Thus we have arrived at a contradiction and hence  $\zeta_0 \in \ell \subset Q$ .  $\square$

*Proof of Theorem 6.1.* It suffices to show that the nonconstant holomorphic mapping

$$f \equiv \frac{\omega_1}{\omega_2}: E \rightarrow \mathbb{P}^1$$

has a nonempty compact level. Assuming  $f$  has no nonempty compact levels, we will apply the above lemmas to obtain a contradiction. For this, we let  $C$  be the set of critical points and we let  $Z$  be the union of all connected components  $A$  of  $C$  for which  $\bar{A}$  is noncompact and meets  $\partial E$ .

According to Lemma 1.1, the levels over almost every point in  $\mathbb{P}^1$  have finite volume. According to Lemma 6.9 and Lemma 6.10, the set

$$Q = \{ \zeta \in \mathbb{P}^1 \mid f^{-1}(\zeta) \Subset X \} = \{ \zeta \in \mathbb{P}^1 \mid L \Subset X \text{ for each level } L \text{ over } \zeta \}$$

is an open set containing  $\mathbb{P}^1 \setminus f(Z)$ . Moreover,  $Z$  has only finitely many connected components, since each connected component must meet the (compact) boundary of any fixed relatively compact neighborhood of  $\partial E$  in  $X$ . Consequently,  $f(Z) \supset \mathbb{P}^1 \setminus Q$  is a finite set and Lemma 6.9 implies that  $Q \neq \mathbb{P}^1$ . Thus  $\mathbb{P}^1 \setminus Q = \{\zeta_1, \dots, \zeta_m\}$  for distinct points  $\zeta_1, \dots, \zeta_m$  and we may choose domains  $U_1, \dots, U_m$  in  $\mathbb{P}^1$  such that, for each  $i = 1, \dots, m$ , we have  $\zeta_i \in U_i$ ,  $\overline{U_i} \cap \overline{U_j} = \emptyset$  for all  $j \neq i$ ,  $0 \in U_i$  if and only if  $\zeta_i = 0$ , and  $\infty \in U_i$  if and only if  $\zeta_i = \infty$ . We may fix a relatively compact neighborhood  $\Omega$  of  $\partial E$  in  $X$  so that  $\Omega$  contains  $f^{-1}(\infty)$  if  $\infty \in Q$ . Lemma 6.9 implies that the set  $H = \overline{\Omega} \cup f^{-1}(\mathbb{P}^1 \setminus (U_1 \cup \dots \cup U_m))$  is compact. Let  $E_0$  be a connected component of  $X \setminus H$  which is contained in  $E$  and which has noncompact closure in  $X$ . We have  $f(E_0) \subset U_i$  for some  $i$  and we may define the holomorphic function  $h: E_0 \rightarrow \mathbb{C}$  by

$$h = \begin{cases} 1/f \upharpoonright_{E_0} & \text{if } \zeta_i = \infty \\ f \upharpoonright_{E_0} - \zeta_i & \text{if } \zeta_i \in \mathbb{C} \end{cases}$$

The inverse image of any compact subset of  $\overline{U_i} \setminus \{\zeta_i\}$  is relatively compact in  $X$ , so  $h$  must vanish at infinity in  $X$ . Proposition 6.6 then implies that some nonempty open subset of  $E_0$  admits a proper holomorphic mapping onto a Riemann surface. In particular,  $f$  has a compact level in  $E_0 \subset E$  and we have arrived at a contradiction. Thus the theorem is proved.  $\square$

*Remark.* If we assume that the end  $E$  is hyperbolic in  $(X, g)$ , then we need only apply Lemma 6.3 in place of Proposition 6.6 in order to obtain a contradiction. So the proof in this case (which is the case required for [NR6]) is simpler.

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